# A short proof of the Banach–Tarski paradox

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Abstract. In 1923, the mathematicians Stefan Banach and Alfred Tarski proved that it was possible to cut a ball in a finite number of pieces and to rearrange the pieces in order to obtain two balls identical in size to the original ball (it is understood that such balls are taken in the Euclidean space $\mathbb{R}^3$).

This result is known as the *Banach–Tarski paradox*. It is related to the *paradox of the sphere* (or *Hausdorff paradox*, named after Felix Hausdorff who discovered it in 1914), whose proof makes use of the Axiom of Choice.

The trick is, of course, that one cannot actually describe the required pieces to carry out such a duplication. Because this incredible result may at first seem apparently false, it was taken as an argument against the validity of the Axiom of Choice. Indeed, the source of the paradox is to be found in the fact that there exist sets in $\mathbb{R}^3$ that are not measurable, which, as is well known, cannot be proven without the Axiom of Choice. The paradox also arises from the strange properties of free groups.

Formally speaking, the result of Banach and Tarski is a theorem of ZFC (that is, of the theory of Zermelo–Fränkel together with the Axiom of Choice), but this will be irrelevant here, and a basic knowledge of naive set theory will be sufficient.

In this article we shall prove the Hausdorff paradox, the Banach–Tarski paradox and its generalisation known as the Banach–Tarski theorem.

# 1 Congruence and equidecomposability

Let $G$ be a group and $X$ a $G$-set. Two subsets $A$ and $B$ of $X$ are said to be *$G$-congruent* if there exists an element $g$ of $G$ such that $gA = B$. It is trivially verified that $G$-congruence is an equivalence relation on $\mathscr{P}(X)$. We shall note $\mathscr{C}_G$ that relation and we shall write $A \overset{\mathscr{C}}{\sim}_G B$ to mean $(A, B) \in \mathscr{C}_G$. A *$G$-congruence map* between $A$ and $B$ is a bijection defined by the action of an element $g$ of $G$.

We say that a finite partition of a set $X$ is a *decomposition* of $X$ (this is not a standard term).

Let $G$ be a group and $X$ a $G$-set. Two subsets $A$ and $B$ of $X$ are said to be *$G$-equidecomposable* if there exist decompositions $\{A_i \mid 1 \leq i \leq n\}$ and $\{B_i \mid 1 \leq i \leq n\}$ of $A$ and $B$ such that for all $i$ in $\{1, \ldots, n\}$, $A_i$ and $B_i$ are $G$-congruent. Again, $G$-equidecomposability is an equivalence relation on $\mathscr{P}(X)$, which we shall denote by $\mathscr{E}_G$. We define a *$G$-puzzle map* between $A$ and $B$ to be a bijective map $f\colon A \to B$ with the property that there exist decompositions $\{A_i \mid 1 \leq i \leq n\}$, $\{B_i \mid 1 \leq i \leq n\}$ of $A$ and $B$ such that for all $i$ in $\{1, \ldots, n\}$ the restriction of $f$ to $A_i$ is a $G$-congruence map between $A_i$ and $B_i$. Decompositions of $A$ and $B$ satisfying the definition of $G$-equidecomposability define and are defined by a $G$-puzzle map between $A$ and $B$.

We say that $A$ is *$G$-subdecomposable* to $B$, and we note $A \precsim_G B$, if there exists a set $B' \subseteq B$ such that $A$ and $B'$ are $G$-equidecomposable.

Theorem 1. *Let $S$ be a set, $R$ a relation on $S$ and $E$ an equivalence relation on $S$. Suppose that the following assertions are verified.*

(*i*) *For any sets $a$ and $b$ in $S$ such that $(a, b) \in R$, there exists an injective map $f\colon a \to b$ such that for any subset $c$ of $a$, $(c, f(c)) \in E$.*

(*ii*) *For any sets $a_1$, $a_2$, $b_1$ and $b_2$ in $S$, if $a_1 \cap a_2 = \varnothing$, $b_1 \cap b_2 = \varnothing$, $(a_1, b_1) \in E$ and $(a_2, b_2) \in E$, then $(a_1 \cup a_2, b_1 \cup b_2) \in E$.*

*Then for all $a$ and $b$ in $S$, $(a, b) \in R$ and $(b, a) \in R$ implies $(a, b) \in E$.*

*Proof.* Let $a$ and $b$ in the set $S$ be such that $(a, b) \in R$ and $(b, a) \in R$, and let $a'$, $b'$ be subsets of $a$ and $b$ respectively with bijections $f\colon a \to b'$ and $g\colon b \to a'$ satisfying the hypothesis $(i)$. We let

$$c = \bigcup_{n \in \mathbb{N}} (g \circ f)^n (a \setminus a').$$

Since $a \setminus c = g(b \setminus f(c))$ (this is a trivial exercise), we find $(a \setminus c, b \setminus f(c)) \in E$; but we also have $(c, f(c)) \in E$, and hence $(a, b)$ is in $E$ by condition $(ii)$. $\square$

(The reader with some knowledge of set theory will see at once that the set $S$ above can be replaced by an arbitrary collection. The Schröder–Bernstein theorem is then an obvious corollary with $S$ being the collection of all sets, $R$ the collection of couples $(a, b)$ such that there exists a injective map from $a$ to $b$ and $E$ the relation of bijection between sets.)

A corollary which applies to our present study is the so-called Banach–Schröder–Bernstein theorem:

PROPOSITION 2. *If $G$ is a group and $X$ is a $G$-set then $G$-subdecomposability is an ordering of $\mathscr{P}(X)$ with respect to $G$-equidecomposability.*

*Proof.* The reflexivity and transitivity of $G$-subdecomposability are easily verified. We contend that the conditions $(i)$ and $(ii)$ of theorem 1 are verified for $G$-subdecomposability and $G$-equidecomposability, which will prove antisymmetry. Indeed, if $A$ and $B$ are subsets of $X$ such that $A \overset{\mathscr{E}}{\sim}_G B'$ for some subset $B'$ of $B$, then a $G$-puzzle map between $A$ and $B'$ satisfies $(i)$. The condition $(ii)$ is obvious given the definition of $G$-equidecomposability. $\square$

Let $G$ be a group and $X$ a $G$-set. We say that a subset $C$ of $X$ is *$G$-paradoxical* if there exist two disjoint subsets $A$ and $B$ of $C$ such that $A$, $B$, and $C$ are pairwise $G$-equidecomposable. We say that a group $G$ is *paradoxical* if it is $G$-paradoxical when viewed as a set on which $G$ operates by translation (on the left).

As a trivial corollary to the Banach–Schröder–Bernstein theorem, a subset $C$ of $X$ can be "duplicated" (that is, there exists a subset $A$ of $C$ such that $A \overset{\mathscr{E}}{\sim}_G C$ and $C \setminus A \overset{\mathscr{E}}{\sim}_G C$) if and only if it is $G$-paradoxical.

Note that if a $G$-set is $H$-paradoxical for some subgroup $H$ of $G$, then it is also $G$-paradoxical.

With these definitions, the Banach–Tarski paradox can be expressed in this way: *the unit ball in* $\mathbb{R}^3$ *is $G$-paradoxical, where* $G = \mathrm{Isom}(\mathbb{R}^3)$ *is the group of isometries of* $\mathbb{R}^3$. In fact, we need only use direct isometries, but this shall play no rôle in the elementary discussion that follows.

This is only a special case of a more general theorem, known as the Banach–Tarski theorem, whose point is that *any* two bounded subsets of $\mathbb{R}^3$ whose interiors are non empty are $\mathrm{Isom}(\mathbb{R}^3)$-equidecomposable.

## 2 Free groups and free actions

We now turn shortly our study to free groups and free actions which play a key rôle in the proof of the Hausdorff paradox. We denote by $F_n$ the free group on $n$ generators.

PROPOSITION 3. *The group $F_2$ is paradoxical.*

*Proof.* We know that any element of $F_n$ can be written in a unique way as a reduced product (or *word*) of its generators and their inverses (the empty word being the unit element). Let $a$ and $b$ be generators of $F_2$. We note $A_+$ (respectively $A_-$, $B_+$ and $B_-$) the subset of $F_2$ consisting of all words starting on the left with $a$ (respectively $a^{-1}$, $b$ and $b^{-1}$). Then $A_- \cup A_+$ and $B_- \cup B_+$ are disjoint subsets of $F_2$ and have decompositions $\{A_-, A_+\}$ and $\{B_-, B_+\}$. Hence $A_- \cup A_+ \overset{\mathscr{E}}{\sim}_{F_2} aA_- \cup A_+ = F_2$ and $B_- \cup B_+ \overset{\mathscr{E}}{\sim}_{F_2} bB_- \cup B_+ = F_2$, as was to be shown. □

The following proposition makes use of the Axiom of Choice.

PROPOSITION 4. *Let $G$ be a group that operates freely on a set $X$. If $G$ is paradoxical, then $X$ is $G$-paradoxical.*

*Proof.* Using the Axiom of Choice, there exists a set $T$ which contains exactly one element of each orbit of $G$ on $X$. If $H_1$ and $H_2$ are subsets of $G$ such that $G$, $H_1$ and $H_2$ are pairwise $G$-equidecomposable, then clearly $G(T)$, $H_1(T)$ and $H_2(T)$ are pairwise $G$-equidecomposable. Since $H_1(T)$ and $H_2(T)$ are disjoint (because the action of $G$ is free) and $G(T) = X$, $X$ is $G$-paradoxical. □

PROPOSITION 5. *Let $G$ be a group and $X$ a $G$-set. Then $G$ operates freely on $X \setminus D$ where $D$ is the set of all the points fixed by a non trivial element of $G$.*

*Proof.* We first prove that $X \setminus D$ is stable under the action of $G$. If $y$ is in $D$ and $g$ is in $G$, and if $h \in G$ is such that $hy = y$, then $hg^{-1}$ fixes $gx$, and hence $gx$ is in $D$. By complementarity, $G(X \setminus D) = X \setminus D$. We now have to show that for any two distinct elements $g$ and $h$ in $G$ and any point $x$ in $X \setminus D$ we have $gx \neq hx$. Suppose that $gy = hy$ for some $y$ in $X$. Then $y = g^{-1}h(y)$, and since we supposed that $g^{-1}h$ is not trivial, $y$ is in $D$. □

# 3 The Hausdorff paradox

We wish to show that the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ is $SO(3)$-paradoxical, where $SO(3)$ is the group of rotations of the Euclidean space $\mathbb{R}^3$. We proceed in two steps. First, we prove that there exists a group of rotations $H$ that is isomorphic to the free group on 2 generators. This implies that $\mathbb{S}^2 \setminus D$ is $SO(3)$-paradoxical (and in particular $\mathrm{Isom}(\mathbb{R}^3)$-paradoxical), where $D$ is the set of points that are fixed by a non trivial rotation of $H$. Since $H$ is denumerable and since a non trivial rotation has exactly two fixed points on the sphere, it follows that the set $D$ is denumerable. In a second step we get rid of the denumerable set $D$ by proving that $\mathbb{S}^2 \setminus D$ and $\mathbb{S}^2$ are $SO(3)$-equidecomposable, which implies at once that the unit sphere is $SO(3)$-paradoxical.

The construction of the free rotation group given below is due to Stanislaw Świerczkowski.

Proposition 6. *There exists a subgroup of $SO(3)$ that is free on 2 generators.*

*Proof.* We define two rotations $u$ and $v$ in $SO(3)$ by their matrices

$$U = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix}$$

with respect to the canonical base of $\mathbb{R}^3$. These are rotations of angle $\arccos(3/5)$ around the $z$-axis and $x$-axis respectively. We shall now prove that any non trivial word consisting of $u$, $u^{-1}$, $v$ and $v^{-1}$ is not equal to the identity, whence $\langle u, v\rangle$ will be free on 2 generators.

Let $w$ be a non trivial word in $\langle u, v\rangle$. Since $uwu^{-1}$ and $u^{-1}wu$ will be the identity if and only if $w$ is the identity, we can suppose without loss of generality that $w$ ends with $u$ or $u^{-1}$. Let us prove that $w(1,0,0) = (a,b,c)/5^k$ for some $a$, $b$ and $c$ in $\mathbb{Z}$ with $b$ not divisible by 5 (in particular $b \neq 0$) and $k$ being the length of the word $w$; this will imply that $w(1,0,0) \neq (1,0,0)$ and thus conclude the proof.

We proceed by induction on the length of the word. If $w$ is a word of length 1, since $w$ is either $u$ or $u^{-1}$, we compute $w(1,0,0) = (3,4,0)/5$, and the claimed result is verified. We now suppose that the result is verified for words of length smaller or equal to $k-1$. We write $w = \nu w'$ where

$\nu$ is either $u$, $u^{-1}$, $v$ or $v^{-1}$ and $w'$ is a word of length $k-1$ such that $w'(1,0,0) = (a',b',c')/5^{k-1}$ where $a'$, $b'$ and $c'$ are integers and $b'$ is not divisible be 5. Then $w(1,0,0) = (a,b,c)/5^k$ where $a = 3a' \mp 4b'$, $b = 3b' \pm 4a'$ and $c = 5c'$ if $\nu = u^{\pm 1}$ or $a = 5a'$, $b = 3b' \mp 4c'$ and $c = 3a' \pm 4b'$ if $\nu = v^{\pm 1}$ respectively. Thus $a$, $b$ and $c$ are integers as we claimed. We still have to show that $b$ is not divisible by 5. For this we write $w' = \nu' w''$ in the same way as before with $w''(1,0,0) = (a'',b'',c'')/5^{k-2}$ and we consider four cases.

(a) If $w$ starts with $u^{\pm 1}v^{\pm 1}$ or $u^{\pm 1}v^{\mp 1}$, then $b = 3b' \pm 4a'$ and 5 divides $a'$ because $a' = 5a''$.

(b) If $w$ starts with $v^{\pm 1}u^{\pm 1}$ or $v^{\pm 1}u^{\mp 1}$, then $b = 3b' \mp 4c'$ and 5 divides $c'$ because $c' = 5c''$.

(c) If $w$ starts with $u^{\pm 2}$, then $b = 3b' \pm 4a' = 3b' \pm (12a'' \mp 16b'') = 3b' + 9b'' \pm 12a'' - 16b'' - 9b'' = 3b' + 3(3b'' \pm 4a'') - 25b'' = 6b' - 25b''$.

(d) If $w$ starts with $v^{\pm 2}$, we find as in (c) $b = 6b' - 25b''$.

Since by the induction hypothesis 5 does not divide $b'$, it follows that in all cases 5 does not divide $b$, and this concludes the proof. □

We have thus proved that $\mathbb{S}^2 \setminus D$ is $SO(3)$-paradoxical for some denumerable set $D$.

*A note on the original proof.* — The proof of the $SO(3)$-paradoxicality of the unit sphere (modulo a denumerable set) by Felix Hausdorff was different and more complicated than the one above; in particular he did not make use of free groups, though the idea of freeness was present. In fact he proved the following: *there exists a decomposition $\{A,B,C,D\}$ of the unit sphere $\mathbb{S}^2$ such that $D$ is denumerable, and $A$, $B$, $C$ and $B \cup C$ are pairwise $SO(3)$-congruent.* From this the reader will easily see that $\mathbb{S}^2 \setminus D$ is $SO(3)$-paradoxical.

To show this he considered the group generated by the rotations $u$ and $v$ whose matrices with respect to the canonical base of $\mathbb{R}^3$ are

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

These are rotations of angles $\pi$ and $-2\pi/3$ around the line $L = \{(x,x,0) \,|\, x \in \mathbb{R}\}$ and the $x$-axis respectively. The group $\langle u, v \rangle$ is obviously not a free group, since $u^2$ and $v^3$ are the identity. But it keeps the interesting property that each of its element can be written in a unique way as a "word" consisting of $u$, $v$ and $v^2$. In fact, the group $\langle u, v\rangle$ is isomorphic to the *modular group* $\Gamma \cong C_2 * C_3$ (where $*$ denotes the product in the category of groups).

Using this fact, one can place the elements of $\langle u, v\rangle$ in three subsets which induce a decomposition $\{A, B, C\}$ of $\mathbb{S}^2 \setminus D$ as desired, where $D$ is, as usual, the denumerable set of points of the sphere that are fixed by a non trivial rotation of $\langle u, v\rangle$. (Alternatively, one can show that any free product of two non trivial groups has $F_2$ as a subgroup, and we can then derive the result as above.)

We now come back to the main proof. The following proposition establishes the $SO(3)$-paradoxicality of the unit sphere $\mathbb{S}^2$.

Proposition 7. *If $S$ is a sphere in $\mathbb{R}^3$ and $D$ is a denumerable subset of $S$, then $S$ and $S \setminus D$ are $SO(3)$-equidecomposable.*

*Proof.* The idea of the proof is to show, using the denumerability of $D$, that there exists a rotation of the sphere that can be applied to $D$ as many times as we want without any point of $D$ ever coming back in $D$. Now if we apply this rotation once more to the set of all the points that can be attained in this way together with $D$, we fall back on the the set itself, but $D$ will have disappeared.

Let $O$ be the centre of the sphere $S$. Since $D$ is denumerable, there exists $d$ in $S$ such that neither $d$ nor its opposite are in $D$; we note $L$ the line containing $O$ and $d$. Let $x$ and $y$ be two points in $D$, and we consider rotations $\rho$ of axis $L$ and such that $\rho^n(x) = y$ for some positive integer $n$. Obviously if $x$ and $y$ are not contained in a plane normal to $L$, then there is no such rotation. Otherwise, since $x$ is not on the axis $L$, there exist exactly $n$ such rotations (namely, if the rotation of angle $\alpha$ maps $x$ to $y$, then the $n$ rotations are those of angles $(\alpha + 2k\pi)/n$ for $k$ in $\{0, \ldots, n-1\}$). Hence the set $R$ of all those rotations for all pairs of points $x$ and $y$ in $D$ and all positive integer $n$ is denumerable. Since the set of all rotations of axis $L$ is *not* denumerable, there exists a rotation $r$ of axis $L$ which is not in $R$, that is such that for all $x$ and $y$ in $D$ and all positive integer $n$ we have $r^n(x) \neq y$. This means that the sets $r^n(D)$ for all non negative integers $n$ are pairwise disjoint.

Let $U$ be the disjoint union of all those sets: $U = \cup_{n\geq 0} r^n(D)$. Then $r(U) = \cup_{n\geq 1} r^n(D) = U \setminus D$, and it follows by definition that $U \overset{\mathscr{C}}{\sim}_{SO(3)} U \setminus D$.

This implies that $S = U \cup (S \setminus U) \overset{\mathscr{E}}{\sim}_{SO(3)} (U \setminus D) \cup (S \setminus U) = S \setminus D$ since the unions are disjoint, and this concludes the proof. $\square$

# 4 The duplication of the ball

From here henceforth, it will be assumed that (when not specified) the group $G$ appearing in the definitions of $G$-congruence, $G$-equidecomposability and $G$-paradoxicality of subsets of $\mathbb{R}^3$ is the group of isometries $\mathrm{Isom}(\mathbb{R}^3)$.

From the paradox of the sphere, we shall deduce a similar result for the unit ball. This is easily done if we ignore at first the centre of the ball.

Indeed, let $A$ and $B$ be disjoint subsets of the unit sphere such that $A$, $B$, and $\mathbb{S}^2$ are pairwise $SO(3)$-equidecomposable. We obtain the $SO(3)$-paradoxicality of the closed unit ball (without its centre) $\mathbb{B}^3 \setminus \{0\}$ if we consider the unions $A'$ and $B'$ of all the segments from 0 (not included) to the points of $A$ and $B$ respectively. Namely, the sets

$$A' = \bigcup_{a \in A} ]0, a], \quad B' = \bigcup_{b \in B} ]0, b]$$

and $\mathbb{B}^3 \setminus \{0\}$ are pairwise $SO(3)$-equidecomposable.

Proposition 8. *Let $B$ be a closed ball of centre $O$. Then $B$ and $B \setminus O$ are equidecomposable.*

*Proof.* We suppose without loss of generality that $B$ is the closed unit ball. We consider the set

$$D = \{(\cos n, \sin n, 0) \mid n \in \mathbb{N}\}.$$

Let $\rho$ be the rotation of 1 radian around the vertical axis. Because no multiple of $2\pi$ is an integer, it is obvious that $\rho(D) = D \setminus \{(1, 0, 0)\}$, whence $B = (B \setminus D) \cup D \overset{\mathscr{E}}{\sim} (B \setminus D) \cup (D \setminus \{(1, 0, 0)\}) = B \setminus \{(1, 0, 0)\}$. Now by using the decomposition $\{B \setminus \{(1, 0, 0), 0\}, \{(1, 0, 0)\}\}$ we find that $B \setminus \{(1, 0, 0)\} \overset{\mathscr{E}}{\sim} B \setminus \{0\}$, and hence by transitivity $B \overset{\mathscr{E}}{\sim} B \setminus \{0\}$, which is the desired result. $\square$

Thus, the closed unit ball $\mathbb{B}^3$ is paradoxical. In particular, there exists a decomposition $\{A, B\}$ of $\mathbb{B}^3$ such that $A$, $B$ and $\mathbb{B}^3$ are pairwise equidecomposable. Using a translation, the set $A$ is also equidecomposable to some other ball of radius 1 disjoint from $\mathbb{B}^3$: we have duplicated the ball. Of course, the Banach–Tarski paradox can be extended by a trivial induction to the following proposition.

PROPOSITION 9. *A finite disjoint union of closed balls of radius* 1 *and the closed unit ball are equidecomposable.*

# 5 The Banach–Tarski theorem

LEMMA 1. *Any closed ball in* $\mathbb{R}^3$ *is equidecomposable to the closed unit ball.*

*Proof.* Let $B$ be a ball of radius $R > 1$ and $\mathcal{B} = \{B_i \mid 1 \leq i \leq n\}$ a covering of $B$ by balls of radius 1. Consider $n$ disjoint balls of radius 1 somewhere else in $\mathbb{R}^3$, say $\mathcal{C} = \{C_i \mid 1 \leq i \leq n\}$. Out of $\mathcal{B}$ we construct a new covering $\mathcal{B}' = \{B'_i \mid 1 \leq i \leq n\}$ of $B$ consisting of disjoint sets in this way: we let $B'_1 = B_1$ and by induction

$$B'_i = B_i \setminus \bigcup_{j=1}^{i-1} B_j$$

whenever $2 \leq i \leq n$. Thus we have $\mathbb{B}^3 \precsim B \precsim \cup_{i=1}^n B'_i \precsim \cup_{i=1}^n C_i \sim \mathbb{B}^3$, hence equidecomposability holds everywhere by the Banach–Schröder–Bernstein theorem.

If $B$ is a ball of radius $R < 1$, we invert the rôles of $B$ and $\mathbb{B}^3$ in the previous proof. $\square$

The above proof can of course be generalised in other situations, but one important aspect is that we were able to select $n$ disjoint balls in $\mathbb{R}^3$.

PROPOSITION 10. *Let* $A$ *be a bounded subset of* $\mathbb{R}^3$ *with non empty interior. Then* $A$ *and the unit ball* $\mathbb{B}^3$ *are equidecomposable.*

*Proof.* There exist reals $r > 0$ and $R > 0$ such that $B(a, r) \subseteq A \subseteq B(0, R)$ for some $a$ in $A$. Moreover by the lemma 1 all balls are equidecomposable to the unit ball, hence $A \precsim B(0, R) \sim \mathbb{B}^3 \sim B(a, r) \precsim A$, and equidecomposability holds everywhere by the Banach–Schröder–Bernstein theorem. $\square$

The Banach–Tarski theorem follows from the transitivity of $\mathscr{E}$:

COROLLARY. *Any two bounded subsets of* $\mathbb{R}^3$ *with empty interiors are equidecomposable.*