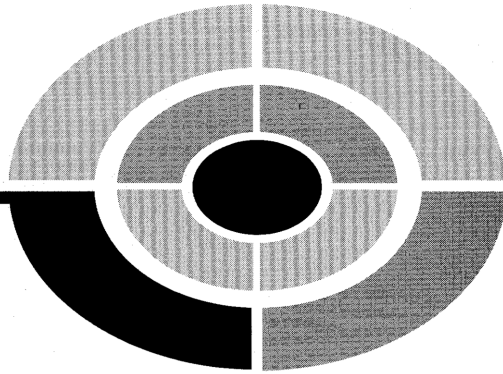


Equations différentielles

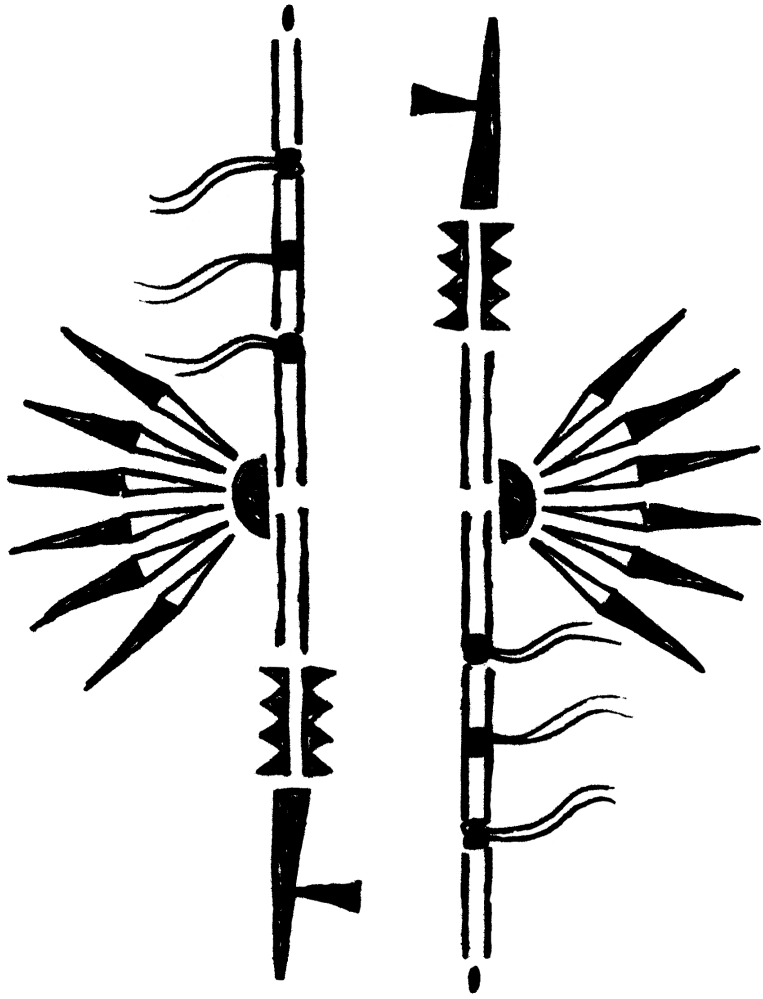
Steven G. KRANTZ

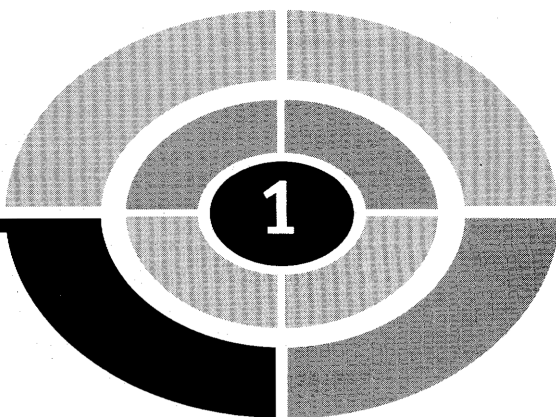
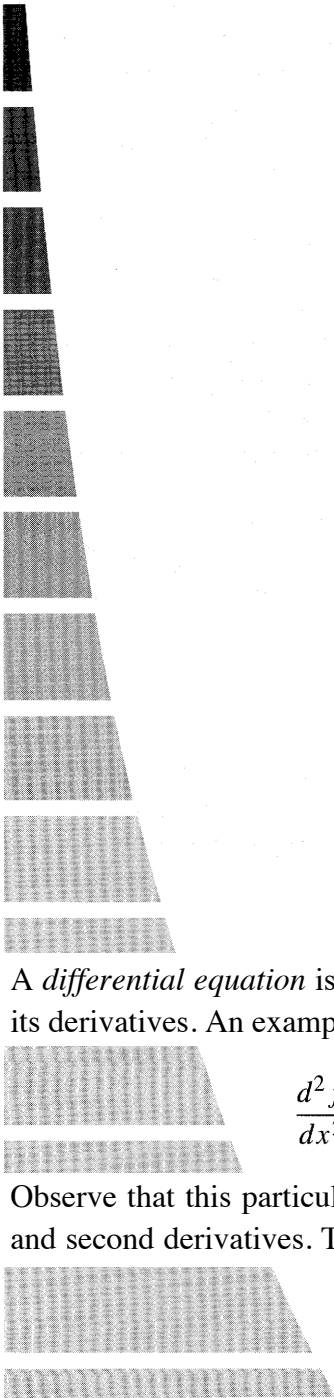
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CHAPTER

1

What Is a Differential Equation?

1.1 Introductory Remarks

A *differential equation* is an equation relating some function f to one or more of its derivatives. An example is

$$\frac{d^2 f}{dx^2} + 2x \frac{df}{dx} + f^2(x) = \sin x. \quad (1)$$

Observe that this particular equation involves a function f together with its first and second derivatives. The objective in solving an equation like (1) is to *find the*



1

function f . Thus we already perceive a fundamental new paradigm: When we solve an algebraic equation, we seek a number or perhaps a collection of numbers; but when we solve a differential equation we seek one or more *functions*.

Many of the laws of nature—in physics, in engineering, in chemistry, in biology, and in astronomy—find their most natural expression in the language of differential equations. Put in other words, differential equations are the language of nature. Applications of differential equations also abound in mathematics itself, especially in geometry and harmonic analysis and modeling. Differential equations occur in economics and systems science and other fields of mathematical science.

It is not difficult to perceive why differential equations arise so readily in the sciences. If $y = f(x)$ is a given function, then the derivative df/dx can be interpreted as the rate of change of f with respect to x . In any process of nature, the variables involved are related to their rates of change by the basic scientific principles that govern the process—that is, by the laws of nature. When this relationship is expressed in mathematical notation, the result is usually a differential equation.

Certainly Newton's Law of Universal Gravitation, Maxwell's field equations, the motions of the planets, and the refraction of light are important physical examples which can be expressed using differential equations. Much of our understanding of nature comes from our ability to solve differential equations. The purpose of this book is to introduce you to some of these techniques.

The following example will illustrate some of these ideas. According to Newton's second law of motion, the acceleration a of a body of mass m is proportional to the total force F acting on the body. The standard implementation of this relationship is

$$F = m \cdot a. \quad (2)$$

Suppose in particular that we are analyzing a falling body of mass m . Express the height of the body from the surface of the Earth as $y(t)$ feet at time t . The only force acting on the body is that due to gravity. If g is the acceleration due to gravity (about -32 ft/sec² near the surface of the Earth) then the force exerted on the body is $m \cdot g$. And of course the acceleration is d^2y/dt^2 . Thus Newton's law (2) becomes

$$m \cdot g = m \cdot \frac{d^2y}{dt^2} \quad (3)$$

or

$$g = \frac{d^2y}{dt^2}.$$

We may make the problem a little more interesting by supposing that air exerts a resisting force proportional to the velocity. If the constant of proportionality is k ,

then the total force acting on the body is $mg - k \cdot (dy/dt)$. Then the equation (3) becomes

$$m \cdot g - k \cdot \frac{dy}{dt} = m \cdot \frac{d^2y}{dt^2}. \quad (4)$$

Equations (3) and (4) express the essential attributes of this physical system.

A few additional examples of differential equations are these:

$$(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + p(p + 1)y = 0; \quad (5)$$

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - p^2)y = 0; \quad (6)$$

$$\frac{d^2y}{dx^2} + xy = 0; \quad (7)$$

$$(1 - x^2)y'' - xy' + p^2y = 0; \quad (8)$$

$$y'' - 2xy' + 2py = 0; \quad (9)$$

$$\frac{dy}{dx} = k \cdot y. \quad (10)$$

Equations (5)–(9) are called Legendre's equation, Bessel's equation, Airy's equation, Chebyshev's equation, and Hermite's equation respectively. Each has a vast literature and a history reaching back hundreds of years. We shall touch on each of these equations later in the book. Equation (10) is the equation of exponential decay (or of biological growth).

Math Note: A great many of the laws of nature are expressed as second-order differential equations. This fact is closely linked to Newton's second law, which expresses force as mass time acceleration (and acceleration is a *second derivative*). But some physical laws are given by higher-order equations. The Euler–Bernoulli beam equation is fourth-order.



Each of equations (5)–(9) is of second-order, meaning that the highest derivative that appears is the second. Equation (10) is of first-order, meaning that the highest derivative that appears is the first. Each equation is an *ordinary differential equation*, meaning that it involves a function of a single variable and the *ordinary derivatives* (not partial derivatives) of that function.



1.2 The Nature of Solutions

An ordinary differential equation of order n is an equation involving an unknown function f together with its derivatives

$$\frac{df}{dx}, \frac{d^2f}{dx^2}, \dots, \frac{d^n f}{dx^n}.$$

We might, in a more formal manner, express such an equation as

$$F\left(x, y, \frac{df}{dx}, \frac{d^2f}{dx^2}, \dots, \frac{d^n f}{dx^n}\right) = 0.$$

How do we verify that a given function f is actually the solution of such an equation?

The answer to this question is best understood in the context of concrete examples.



EXAMPLE 1.1

Consider the differential equation

$$y'' - 5y' + 6y = 0.$$

Without saying how the solutions are actually *found*, we can at least check that $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ are both solutions.

To verify this assertion, we note that

$$\begin{aligned} y_1'' - 5y_1' + 6y_1 &= 2 \cdot 2 \cdot e^{2x} - 5 \cdot 2 \cdot e^{2x} + 6 \cdot e^{2x} \\ &= [4 - 10 + 6] \cdot e^{2x} \\ &\equiv 0 \end{aligned}$$

and

$$\begin{aligned} y_2'' - 5y_2' + 6y_2 &= 3 \cdot 3 \cdot e^{3x} - 5 \cdot 3 \cdot e^{3x} + 6 \cdot e^{3x} \\ &= [9 - 15 + 6] \cdot e^{3x} \\ &\equiv 0. \end{aligned}$$

This process, of verifying that a *function* is a solution of the given differential equation, is most likely entirely new for you. You will want to practice and become accustomed to it. In the last example, you may check that any function of the form

$$y(x) = c_1 e^{2x} + c_2 e^{3x} \tag{1}$$

(where c_1, c_2 are arbitrary constants) is also a solution of the differential equation.

Math Note: This last observation is an instance of the principle of superposition in physics. Mathematicians refer to the algebraic operation in equation (1) as “taking a linear combination of solutions” while physicists think of the process as superimposing forces.



An important obverse consideration is this: When you are going through the procedure to solve a differential equation, how do you know when you are finished? The answer is that the solution process is complete when all derivatives have been eliminated from the equation. For then you will have y expressed in terms of x (at least implicitly). Thus you will have found the sought-after function.

For a large class of equations that we shall study in detail in the present book, we will find a number of “independent” solutions equal to the order of the differential equation. Then we will be able to form a so-called “general solution” by combining them as in (1). Of course we shall provide all the details of this process in the development below.

You Try It: Verify that each of the functions $y_1(x) = e^x$, $y_2(x) = e^{2x}$ and $y_3(x) = e^{-4x}$ is a solution of the differential equation



$$\frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} - 10 \frac{dy}{dx} + 8y = 0.$$

More generally, check that $y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{-4x}$ (where c_1, c_2, c_3 are arbitrary constants) is a “general solution” of the differential equation.

Sometimes the solution of a differential equation will be expressed as an *implicitly defined function*. An example is the equation

$$\frac{dy}{dx} = \frac{y^2}{1 - xy}, \quad (2)$$

which has solution

$$xy = \ln y + c. \quad (3)$$

Equation (3) represents a solution because all derivatives have been eliminated.

Example 1.2 below contains the details of the verification that (3) is the solution of (2).

Math Note: It takes some practice to get used to the idea that an implicitly defined function is still a function. A classic and familiar example is the equation



$$x^2 + y^2 = 1. \quad (4)$$

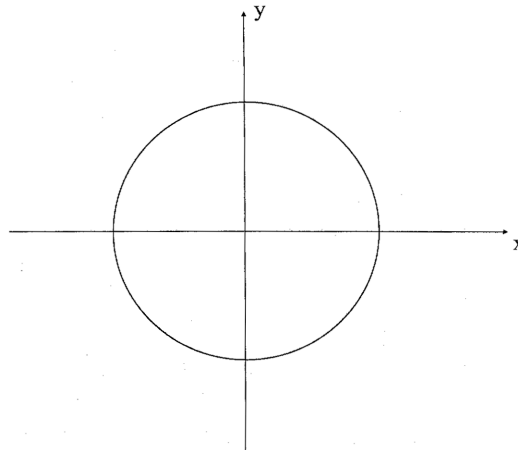


Fig. 1.1.

This relation expresses y as a function of x at most points. Refer to Fig. 1.1. In fact the equation (4) entails

$$y = +\sqrt{1 - x^2}$$

when y is positive and

$$y = -\sqrt{1 - x^2}$$

when y is negative. It is only at the exceptional points $(-1, 0)$ and $(1, 0)$, where the tangent lines are vertical, that y *cannot* be expressed as a function of x .

Note here that the hallmark of what we call a *solution* is that it has no derivatives in it: it is a straightforward formula, relating y (the dependent variable) to x (the independent variable).

e.g.

EXAMPLE 1.2

To verify that (3) is indeed a solution of (2), let us differentiate:

$$\frac{d}{dx}[xy] = \frac{d}{dx}[\ln y + c],$$

hence

$$1 \cdot y + x \cdot \frac{dy}{dx} = \frac{dy/dx}{y}$$

or

$$\frac{dy}{dx} \left[\frac{1}{y} - x \right] = y.$$

In conclusion,

$$\frac{dy}{dx} = \frac{y^2}{1 - xy},$$

as desired.

One unifying feature of the two examples that we have now seen of verifying solutions is this: When we solve an equation of order n , we expect n “independent solutions” (we shall have to say later just what this word “independent” means) and we expect n undetermined constants. In the first example, the equation was of order 2 and the undetermined constants were c_1 and c_2 . In the second example, the equation was of order 1 and the undetermined constant was c .

You Try It: Verify that the equation $x \sin y = \cos y$ gives an implicit solution to the differential equation 

$$\frac{dy}{dx} [x \cot y + 1] = -1.$$

1.3 Separable Equations

In this section we shall encounter our first general class of equations with the property that

- (i) We can immediately recognize members of this class of equations.
- (ii) We have a simple and direct method for (in principle)¹ solving such equations.

This is the class of *separable equations*.

DEFINITION 1.1

An ordinary differential equation is *separable* if it is possible, by elementary algebraic manipulation, to arrange the equation so that all the dependent variables (usually the y variable) are on one side and all the independent variables

¹We throw in this caveat because it can happen, and frequently does happen, that we can write down integrals that represent solutions of our differential equation, but we are *unable to evaluate those integrals*. This is annoying, but we shall later—in Chapter 7—learn numerical techniques that will address such an impasse.

(usually the x variable) are on the other side. The corresponding solution technique is called *separation of variables*.

Let us learn the method by way of some examples.

e.g.

EXAMPLE 1.3

Solve the ordinary differential equation

$$y' = 2xy.$$

SOLUTION

In the method of separation of variables—which is a method for *first-order* equations only—it is useful to write the derivative using Leibniz notation. Thus we have

$$\frac{dy}{dx} = 2xy.$$

We rearrange this equation as

$$\frac{dy}{y} = 2x \, dx.$$

[It should be noted here that we use the shorthand dy to stand for $\frac{dy}{dx} dx$.]

Now we can integrate both sides of the last displayed equation to obtain

$$\int \frac{dy}{y} = \int 2x \, dx.$$

We are fortunate in that both integrals are easily evaluated. We obtain

$$\ln y = x^2 + c.$$

[It is important here that we include the constant of integration. We combine the constant from the left-hand integral and the constant from the right-hand integral into a single constant c .] Thus

$$y = e^{x^2+c}.$$

We may abbreviate e^c by D and rewrite this last equation as

$$y = De^{x^2}. \quad (1)$$

Notice two important features of our final representation for the solution:

- (i) We have re-expressed the constant e^C as the positive constant D .
- (ii) Our solution contains one free constant, as we may have anticipated since the differential equation is of order 1.

We invite you to verify that the solution in equation (1) actually satisfies the original differential equation.

EXAMPLE 1.4

Solve the differential equation

e.g.

$$xy' = (1 - 2x^2) \tan y.$$

SOLUTION

We first write the equation in Leibniz notation. Thus

$$x \cdot \frac{dy}{dx} = (1 - 2x^2) \tan y.$$

Separating variables, we find that

$$\cot y \, dy = \left[\frac{1}{x} - 2x \right] dx.$$

Applying the integral to both sides gives

$$\int \cot y \, dy = \int \left[\frac{1}{x} - 2x \right] dx$$

or

$$\ln \sin y = \ln x - x^2 + C.$$

Again note that we were careful to include a constant of integration.

We may express our solution as

$$\sin y = e^{\ln x - x^2 + C}$$

or

$$\sin y = D \cdot x \cdot e^{-x^2}.$$

The result may be written as

$$y = \sin^{-1} \left[D \cdot x \cdot e^{-x^2} \right].$$

We invite you to verify that this is indeed a solution to the given differential equation.



Math Note: It should be stressed that not all ordinary differential equations are separable. As an instance, the equation

$$x^2y + y^2x = \sin(xy)$$

cannot be separated so that all the x 's are on one side of the equation and all the y 's on the other side.



You Try It: Use the method of separation of variables to solve the differential equation

$$x^3y' = y.$$

1.4 First-Order Linear Equations

Another class of differential equations that is easily recognized and readily solved (at least in principle) is that of first-order linear equations.

DEFINITION 1.2

An equation is said to be *first-order linear* if it has the form

$$y' + a(x)y = b(x). \quad (1)$$

The “first-order” aspect is obvious: only first derivatives appear in the equation. The “linear” aspect depends on the fact that the left-hand side involves a differential operator that acts linearly on the space of differentiable functions. Roughly speaking, a differential equation is linear if y and its derivatives are not multiplied together, not raised to powers, and do not occur as the arguments of functions. This is an advanced idea that we shall explicate in detail later. For now, you should simply accept that an equation of the form (1) is first-order linear, and that we will soon have a recipe for solving it.

As usual, we explicate the method by proceeding directly to the examples.



EXAMPLE 1.5

Consider the differential equation

$$y' + 2xy = x.$$

Find a complete solution.

SOLUTION

This equation is plainly not separable (try it and convince yourself that this is so). Instead we endeavor to multiply both sides of the equation by some function that will make each side readily integrable. It turns out that there is a trick that always works: You multiply both sides by $e^{\int a(x) dx}$.

Like many tricks, this one may seem unmotivated. But let us try it out and see how it works in practice. Now

$$\int a(x) dx = \int 2x dx = x^2.$$

[At this point we *could* include a constant of integration, but it is not necessary.] Thus $e^{\int a(x) dx} = e^{x^2}$. Multiplying both sides of our equation by this factor gives

$$e^{x^2} \cdot y' + e^{x^2} \cdot 2xy = e^{x^2} \cdot x$$

or

$$\left[e^{x^2} \cdot y \right]' = x \cdot e^{x^2}.$$

It is the last step that is a bit tricky. For a first-order linear equation, it is *guaranteed* that if we multiply through by $e^{\int a(x) dx}$ then the left-hand side of the equation will end up being the derivative of $[e^{\int a(x) dx} \cdot y]$. Now of course we integrate both sides of the equation:

$$\int \left[e^{x^2} \cdot y \right]' dx = \int x \cdot e^{x^2} dx.$$

We can perform both the integrations: on the left-hand side we simply apply the fundamental theorem of calculus; on the right-hand side we do the integration. The result is

$$e^{x^2} \cdot y = \frac{1}{2} \cdot e^{x^2} + C$$

or

$$y = \frac{1}{2} + Ce^{-x^2}.$$

Observe that, as we usually expect, the solution has one free constant (because the original differential equation was of order 1). We invite you to check that this solution actually satisfies the differential equation.



Math Note: Of course not all ordinary differential equations are first order linear. The equation

$$[y']^2 - y = \sin x$$

is indeed first order—because the highest derivative that appears is the first derivative. But it is *nonlinear* because the function y' is multiplied by itself. The equation

$$y'' \cdot y - y' = e^x$$

is second order and is also nonlinear—because y'' is multiplied times y .

Summary of the method of first-order linear equations

To solve a first-order linear equation

$$y' + a(x)y = b(x),$$

multiply both sides of the equation by the “integrating factor” $e^{\int a(x) dx}$ and then integrate.

e.g.

EXAMPLE 1.6

Solve the differential equation

$$x^2 y' + xy = x^2 \cdot \sin x.$$

SOLUTION

First observe that this equation is not in the standard form (equation (1)) for first-order linear. We render it so by multiplying through by a factor of $1/x^2$. Thus the equation becomes

$$y' + \frac{1}{x}y = \sin x.$$

Now $a(x) = 1/x$, $\int a(x) dx = \ln |x|$, and $e^{\int a(x) dx} = |x|$. We multiply the differential equation through by this factor. In fact, in order to simplify the calculus, we shall restrict attention to $x > 0$. Thus we may eliminate the absolute value signs.

Thus

$$xy' + y = x \cdot \sin x.$$

Now, as is guaranteed by the theory, we may rewrite this equation as

$$[x \cdot y]' = x \cdot \sin x.$$

Applying the integral to both sides gives

$$\int [x \cdot y]' dx = \int x \cdot \sin x dx.$$

As usual, we may use the fundamental theorem of calculus on the left, and we may apply integration by parts on the right. The result is

$$x \cdot y = -x \cdot \cos x + \sin x + C.$$

We finally find that our solution is

$$y = -\cos x + \frac{\sin x}{x} + \frac{C}{x}.$$

You should plug this answer into the differential equation and check that it works.

You Try It: Use the method of first-order linear equations to find the complete solution of the differential equation 

$$y' + \frac{1}{x}y = e^x.$$

1.5 Exact Equations

A great many first-order equations may be written in the form

$$M(x, y) dx + N(x, y) dy = 0. \quad (1)$$

This particular format is quite suggestive, for it brings to mind a family of curves. Namely, if it happens that there is a function $f(x, y)$ so that

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N, \quad (2)$$

then we can rewrite the differential equation as

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0. \quad (3)$$

Of course the only way that such an equation can hold is if

$$\frac{\partial f}{\partial x} \equiv 0 \quad \text{and} \quad \frac{\partial f}{\partial y} \equiv 0.$$

And this entails that the function f be identically constant. In other words,

$$f(x, y) \equiv c.$$

This last equation describes a family of curves: for each fixed value of c , the equation expresses y implicitly as a function of x , and hence gives a curve. In later parts of this book we shall learn much from thinking of the set of solutions of a differential equation as a smoothly varying family of curves in the plane.

The method of solution just outlined is called the *method of exact equations*. It depends critically on being able to tell when an equation of the form (1) can be written in the form (3). This in turn begs the question of when (2) will hold.

Fortunately, we learned in calculus a complete answer to this question. Let us review the key points. First note that, if it is the case that

$$\frac{\partial f}{\partial x} = M \quad \text{and} \quad \frac{\partial f}{\partial y} = N, \quad (4)$$

then we see (by differentiation) that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Since mixed partials of a smooth function may be taken in any order, we find that a *necessary condition* for the condition (4) to hold is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \quad (5)$$

We call (5) the *exactness condition*. This provides us with a useful test for when the method of exact equations will apply.

It turns out that condition (5) is also sufficient—at least on a domain with no holes. We refer you to any good calculus book (see, for instance, [STE]) for the details of this assertion. We will use our worked examples to illustrate the point.

e.g.

EXAMPLE 1.7

Use the method of exact equations to solve

$$\frac{x}{2} \cdot \cot y \cdot \frac{dy}{dx} = -1.$$

SOLUTION

First, we rearrange the equation as

$$2x \sin y \, dx + x^2 \cos y \, dy = 0.$$

Observe that the role of $M(x, y)$ is played by $2x \sin y$ and the role of $N(x, y)$ is played by $x^2 \cos y$. Next we see that

$$\frac{\partial M}{\partial y} = 2x \cos y = \frac{\partial N}{\partial x}.$$

Thus our necessary condition for the method of exact equations to work is satisfied. We shall soon see that it is also sufficient.

We seek a function f such that $\partial f / \partial x = M(x, y) = 2x \sin y$ and $\partial f / \partial y = N(x, y) = x^2 \cos y$. Let us begin by concentrating on the first of these conditions:

$$\frac{\partial f}{\partial x} = 2x \sin y,$$

hence

$$\int \frac{\partial f}{\partial x} \, dx = \int 2x \sin y \, dx.$$

The left-hand side of this equation may be evaluated with the fundamental theorem of calculus. Treating x and y as independent variables (which is part of this method), we can also compute the integral on the right. The result is

$$f(x, y) = x^2 \sin y + \phi(y). \quad (6)$$

Now there is an important point that must be stressed. You should by now have expected a constant of integration to show up. But in fact our “constant of integration” is $\phi(y)$. This is because our integral was with respect to x , and therefore our constant of integration should be the most general possible expression *that does not depend on x* . That, of course, would be a function of y .

Now we differentiate both sides of (6) with respect to y to obtain

$$N(x, y) = \frac{\partial f}{\partial y} = x^2 \cos y + \phi'(y).$$

But of course we already know that $N(x, y) = x^2 \cos y$. The upshot is that

$$\phi'(y) = 0$$

or

$$\phi(y) = d,$$

an ordinary constant.

Plugging this information into equation (6) now yields that

$$f(x, y) = x^2 \sin y + d.$$

We stress that *this is not the solution of the differential equation*. Before you proceed, please review the outline of the method of exact equations that preceded this example. Our job now is to set

$$f(x, y) = c.$$

So

$$x^2 \cdot \sin y = \tilde{c}, \quad (7)$$

where $\tilde{c} = c - d$.

Equation (7) is in fact the solution of our differential equation, expressed implicitly. If we wish, we can solve for y in terms of x to obtain

$$y = \sin^{-1} \frac{\tilde{c}}{x^2}.$$

And you may check that this is the solution of the given differential equation.

e.g.

EXAMPLE 1.8

Use the method of exact equations to solve the differential equation

$$y^2 dx - x^2 dy = 0.$$

SOLUTION

We first test the exactness condition:

$$\frac{\partial M}{\partial y} = 2y \neq -2x = \frac{\partial N}{\partial x}.$$

The exactness condition fails. As a result, this ordinary differential equation cannot be solved by the method of exact equations.

Notice that we are *not* saying here that the given differential equation cannot be solved. In fact it can be solved by the method of separation of variables (try it!). Rather, it cannot be solved by the method of exact equations.

Math Note: It is an interesting fact that the concept of exactness is closely linked to the geometry of the domain of the functions being studied. An important example is



$$M(x, y) = \frac{-y}{x^2 + y^2}, \quad N(x, y) = \frac{x}{x^2 + y^2}.$$

We take the domain of M and N to be $U = \{(x, y) : 1 < x^2 + y^2 < 2\}$ in order to avoid the singularity at the origin. Of course this domain has a hole.

Then you may check that $\partial M/\partial y = \partial N/\partial x$ on U . But it can be shown that there is no function $f(x, y)$ such that $\partial f/\partial x = M$ and $\partial f/\partial y = N$. Again, the hole in the domain is the enemy.

Without advanced techniques at our disposal, it is best when using the method of exact equations to work only on domains that have no holes.

Math Note: It is a fact that, even when a differential equation fails the “exact equations test,” it is always possible to multiply the equation through by an “integrating factor” so that it *will* pass the exact equations test. As an example, the differential equation



$$2xy \sin x \, dx + x^2 \sin x \, dy = 0$$

is *not* exact. But multiply through by the integrating factor $1/\sin x$ and the new equation

$$2xy \, dx + x^2 \, dy = 0$$

is exact.

Unfortunately, it can be quite difficult to discover explicitly what that integrating factor might be. We will learn more about the method of integrating factors in Section 1.8.

EXAMPLE 1.9

Use the method of exact equations to solve

$$e^y \, dx + (xe^y + 2y) \, dy = 0.$$

e.g.

SOLUTION

First we check for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[e^y] = e^y = \frac{\partial}{\partial x}[xe^y + 2y] = \frac{\partial M}{\partial x}.$$

Thus the equation passes the test and the method of exact equations is at least feasible.

Now we can proceed to solve for f :

$$\frac{\partial f}{\partial x} = M = e^y,$$

hence

$$f(x, y) = x \cdot e^y + \phi(y).$$

But then

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} [x \cdot e^y + \phi(y)] = x \cdot e^y + \phi'(y).$$

And this last expression must equal $N(x, y) = xe^y + 2y$. It follows that

$$\phi'(y) = 2y$$

or

$$\phi(y) = y^2 + d.$$

Altogether, then, we conclude that

$$f(x, y) = x \cdot e^y + y^2 + d.$$

We must not forget the final step. The solution of the differential equation is

$$f(x, y) = c$$

or

$$x \cdot e^y + y^2 + d = c$$

or

$$x \cdot e^y + y^2 = \tilde{c}.$$

This time we must content ourselves with the solution expressed implicitly, since it is not feasible to solve for y in terms of x .



You Try It: Use the method of exact equations to solve the differential equation

$$3x^2y \, dx + x^3 \, dy = 0.$$





1.7 Homogeneous Equations

You should be cautioned that the word “homogeneous” has two meanings in this subject (as mathematics is developed simultaneously by many people all over the world, and they do not always stop to cooperate on their choices of terminology).

One usage, which we shall see and use frequently later in the book, is that an ordinary differential equation is *homogeneous* when the right-hand side is zero; that is, there is no forcing term.

The other usage will be relevant to the present section. It bears on the “balance” of weight among the different variables. It turns out that a differential equation in which the x and y variables have a balanced presence is amenable to a useful change of variables. That is what we are about to learn.

First of all, a function $g(x, y)$ of two variables is said to be *homogeneous of degree α* , for α a real number, if

$$g(tx, ty) = t^\alpha g(x, y) \quad \text{for all } t > 0.$$

As examples, consider:

- Let $g(x, y) = x^2 + xy$. Then $g(tx, ty) = t^2 \cdot g(x, y)$, so g is homogeneous of degree 2.
- Let $g(x, y) = \sin[x/y]$. Then $g(tx, ty) = g(x, y) = t^0 \cdot g(x, y)$, so g is homogeneous of degree 0.
- Let $g(x, y) = \sqrt{x^2 + y^2}$. Then $g(tx, ty) = t \cdot g(x, y)$, so g is homogeneous of degree 1.

In case a differential equation has the form

$$M(x, y) dx + N(x, y) dy = 0$$

and M, N have the *same degree of homogeneity*, then it is possible to perform the change of variable $z = y/x$ and make the equation separable (see Section 1.3). Of course we then have a well-understood method for solving the equation.

The next examples will illustrate the method.

EXAMPLE 1.11

Use the method of homogeneous equations to solve the equation

$$(x + y) dx - (x - y) dy = 0.$$

e.g.

SOLUTION

First notice that the equation is *not exact*, so we must use some other method to find a solution. Now observe that $M(x, y) = x + y$ and $N(x, y) = -(x - y)$ and each is homogeneous of degree 1. We thus rewrite the equation in the form

$$\frac{dy}{dx} = \frac{x + y}{x - y}.$$

Dividing numerator and denominator by x , we finally have

$$\frac{dy}{dx} = \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}}. \quad (1)$$

The point of these manipulations is that the right-hand side is now plainly homogeneous of degree 0. We introduce the change of variable

$$z = \frac{y}{x}, \quad (2)$$

hence

$$y = zx$$

and

$$\frac{dy}{dx} = z + x \cdot \frac{dz}{dx}. \quad (3)$$

Putting (2) and (3) into (1) gives

$$z + x \frac{dz}{dx} = \frac{1+z}{1-z}.$$

Of course this may be rewritten as

$$x \frac{dz}{dx} = \frac{1+z^2}{1-z}$$

or

$$\frac{1-z}{1+z^2} dz = \frac{dx}{x}.$$

We apply the integral, and rewrite the left-hand side, to obtain

$$\int \frac{dz}{1+z^2} - \int \frac{z dz}{1+z^2} = \int \frac{dx}{x}.$$

The integrals are easily evaluated, and we find that

$$\tan^{-1} z - \frac{1}{2} \ln(1+z^2) = \ln x + C.$$

Now we return to our original notation by setting $z = y/x$. The result is

$$\tan^{-1} \frac{y}{x} - \ln \sqrt{x^2 + y^2} = C.$$

Thus we have expressed y implicitly as a function of x , and thereby solved the differential equation.



Math Note: Of course it should be clearly understood that most functions are *not* homogeneous. The functions

- $f(x, y) = x + y^2$
- $f(x, y) = x \sin y$
- $f(x, y) = e^{xy}$
- $f(x, y) = \log(x^2 y)$

have no homogeneity properties.

EXAMPLE 1.12

Solve the differential equation

$$xy' = 2x + 3y.$$

SOLUTION

It is plain that the equation is first-order linear, and we encourage the reader to solve the equation by that method for practice and comparison purposes. Instead, developing the ideas of the present section, we will use the method of homogeneous equations.

If we rewrite the equation as

$$-(2x + 3y) dx + x dy = 0,$$

then we see that each of $M = -(2x + 3y)$ and $N = x$ is homogeneous of degree 1. Thus we have as

$$\frac{dy}{dx} = \frac{2x + 3y}{x}.$$

The right-hand side is homogeneous of degree 0, as we expect.

We set $z = y/x$ and $dy/dx = z + x[dz/dx]$. The result is

$$z + x \cdot \frac{dz}{dx} = 2 + 3\frac{y}{x} = 2 + 3z.$$

The equation separates, as we anticipate, into

$$\frac{dz}{2 + 2z} = \frac{dx}{x}.$$

This is easily integrated to yield

$$\frac{1}{2} \ln(1 + z) = \ln x + C$$

or

$$z = Dx^2 - 1.$$

Resubstituting $z = y/x$ gives

$$\frac{y}{x} = Dx^2 - 1,$$

hence

$$y = Dx^3 - x.$$

We encourage you to check that this is indeed the solution of the given differential equation.

e.g.



You Try It: Use the method of homogeneous equations to solve the differential equation

$$(y^2 - x^2) dx + xy dy = 0.$$

1.8 Integrating Factors

We used a special type of integrating factor in Section 1.4 on first-order linear equations. At that time, we suggested that integrating factors may be applied in some generality to the solution of first-order differential equations. The trick is in *finding* the integrating factor.

In this section we shall discuss this matter in some detail, and indicate the uses and the limitations of the method of integrating factors.

First let us illustrate the concept of integrating factor by way of a concrete example.

EXAMPLE 1.13

The differential equation

$$y dx + (x^2 y - x) dy = 0 \quad (1)$$

is plainly *not exact*, just because $\partial M/\partial y = 1$ while $\partial N/\partial x = 2xy - 1$, and these are unequal. However, if we multiply the equation (1) through by a factor of $1/x^2$, then we obtain the equivalent equation

$$\frac{y}{x^2} dx + \left(y - \frac{1}{x} \right) dy = 0,$$

and this equation is *exact* (as you may easily verify by calculating $\partial M/\partial y$ and $\partial N/\partial x$). And of course we have a direct method (see Section 1.5) for solving such an exact equation.

We call the function $1/x^2$ in this last example an *integrating factor*. It is obviously a matter of some interest to be able to find an integrating factor for any given first-order equation. So, given a differential equation

$$M(x, y) dx + N(x, y) dy = 0,$$

we wish to find a function $\mu(x, y)$ such that

$$\mu(x, y) \cdot M(x, y) dx + \mu(x, y) \cdot N(x, y) dy = 0$$

is exact. This entails

$$\frac{\partial(\mu \cdot M)}{\partial y} = \frac{\partial(\mu \cdot N)}{\partial x}.$$

Writing this condition out, we find that

$$\mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}.$$

This last equation may be rewritten as

$$\frac{1}{\mu} \left(N \frac{\partial \mu}{\partial x} - M \frac{\partial \mu}{\partial y} \right) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}.$$

Now we use the method of wishful thinking: we suppose not only that an integrating factor μ exists, but in fact that one exists that only depends on the variable x (and not at all on y). Then the last equation reduces to

$$\frac{1}{\mu} \frac{d\mu}{dx} = \frac{\partial M/\partial y - \partial N/\partial x}{N}.$$

Notice that the left-hand side of this new equation is a function of x only. Hence so is the right-hand side. Call the right-hand side $g(x)$. Notice that g is something that we can always compute.

Thus

$$\frac{1}{\mu} \frac{d\mu}{dx} = g(x),$$

hence

$$\frac{d(\ln \mu)}{dx} = g(x)$$

or

$$\ln \mu = \int g(x) dx.$$

We conclude that, in case there is an integrating factor μ that depends on x only, then

$$\mu(x) = e^{\int g(x) dx},$$

where

$$g(x) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$$

can always be computed directly from the original differential equation.

Of course the best way to understand a new method like this is to look at some examples. This we now do.

 EXAMPLE 1.14

Solve the differential equation

$$(xy - 1)dx + (x^2 - xy)dy = 0.$$

SOLUTION

You may plainly check that this equation is not exact. It is also not separable. So we shall seek an integrating factor that depends only on x . Now

$$g(x) = \frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{[x] - [2x - y]}{x^2 - xy} = -\frac{1}{x}.$$

This g depends only on x , signaling that the methodology we just developed will actually work.

We set

$$\mu(x) = e^{\int g(x) dx} = e^{\int -1/x dx} = \frac{1}{x}.$$

This is our integrating factor. We multiply the original differential equation through by $1/x$ to obtain

$$\left(y - \frac{1}{x}\right)dx + (x - y)dy = 0.$$

You may check that *this* equation is certainly exact. We omit the details of solving this exact equation, since that methodology was covered in Section 1.5.

Of course the roles of y and x may be reversed in our reasoning for finding an integrating factor. In case the integrating factor μ depends only on y (and not at all on x) then we set

$$h(y) = -\frac{\partial M/\partial y - \partial N/\partial x}{M}$$

and define

$$\mu(y) = e^{\int h(y) dy}.$$

 EXAMPLE 1.15

Solve the differential equation

$$y dx + (2x - ye^y) dy = 0.$$

SOLUTION

First observe that the equation is not exact as it stands. Second,

$$\frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{-1}{2x - ye^y}$$

does *not* depend only on x . So instead we look at

$$-\frac{\partial M/\partial y - \partial N/\partial x}{M} = -\frac{-1}{y},$$

and this expression depends only on y . So it will be our $h(y)$. We set

$$\mu(y) = e^{\int h(y) dy} = e^{\int 1/y dy} = y.$$

Multiplying the differential equation through by $\mu(y) = y$, we obtain the new equation

$$y^2 dx + (2xy - y^2 e^y) dy = 0.$$

You may easily check that this new equation is exact, and then solve it by the method of Section 1.5.

You Try It: Use the method of integrating factors to transform the differential equation 

$$\frac{2y}{x^2} dx + \frac{1}{x} dy = 0$$

to an exact equation. Then solve it.

Math Note: We conclude this section by noting that the differential equation

$$xy^3 dx + yx^2 dy = 0$$

has the properties that

- It is not exact;
- $\frac{\partial M/\partial y - \partial N/\partial x}{N}$ does not depend on x only;
- $-\frac{\partial M/\partial y - \partial N/\partial x}{M}$ does not depend on y only.

Thus the method of the present section is not a panacea. We shall not always be able to find an integrating factor. Still, the technique has its uses.

1.9 Reduction of Order

Later in the book, we shall learn that virtually *any* ordinary differential equation can be transformed to a first-order *system* of equations. This is, in effect, just a notational trick, but it emphasizes the centrality of first-order equations and systems. In the present section, we shall learn how to reduce certain higher-order equations to first-order equations—ones which we can frequently solve.

In each differential equation in this section, x will be the independent variable and y the dependent variables. So a typical second-order equation will involve x, y, y', y'' . The key to the success of each of the methods that we shall introduce in this section is that one variable must be missing from the equation.

1.9.1 DEPENDENT VARIABLE MISSING

In case the variable y is missing from our differential equation, we make the substitution $y' = p$. This entails $y'' = p'$. Thus the differential equation is reduced to first-order.



EXAMPLE 1.16

Solve the differential equation

$$xy'' - y' = 3x^2$$

using reduction of order.

SOLUTION

We set $y' = p$ and $y'' = p'$, so that the equation becomes

$$xp' - p = 3x^2.$$

Observe that this new equation is first-order linear in the new dependent variable p . We write it in standard form as

$$p' - \frac{1}{x}p = 3x.$$

We may solve this equation by using the integrating factor $\mu(x) = e^{\int -1/x dx} = 1/x$. Thus

$$\frac{1}{x}p' - \frac{1}{x^2}p = 3$$

so

$$\left[\frac{1}{x}p\right]' = 3$$

or

$$\int \left[\frac{1}{x}p\right]' dx = \int 3 dx.$$

Performing the integrations, we conclude that

$$\frac{1}{x}p = 3x + C,$$

hence

$$p(x) = 3x^2 + Cx.$$

Now we recall that $p = y'$, so we make that substitution. The result is

$$y' = 3x^2 + Cx,$$

hence

$$y = x^3 + \frac{C}{2}x^2 + D = x^3 + Ex^2 + D.$$

We invite you to confirm that this is the complete and general solution to the original differential equation.

EXAMPLE 1.17

Find the solution of the differential equation

e.g.

$$[y']^2 = x^2 y''.$$

SOLUTION

We note that y is missing, so we make the substitution $p = y'$, $p' = y''$. Thus the equation becomes

$$p^2 = x^2 p'.$$

This equation is amenable to separation of variables.

The result is

$$\frac{dx}{x^2} = \frac{dp}{p^2},$$

which integrates to

$$-\frac{1}{x} = -\frac{1}{p} + E$$

or

$$p = \frac{1}{E} \left(1 - \frac{1}{1 + Ex} \right)$$

for some unknown constant E . We re-substitute $p = y'$ and integrate to obtain finally that

$$y(x) = \frac{x}{E} - \frac{1}{E^2} \ln(1 + Ex) + D$$

is the general solution of the original differential equation.



Math Note: As usual, notice that the solution of any of our second-order differential equations gives rise to two undetermined constants. Usually these will be specified by two initial conditions.



You Try It: Use the method of reduction of order to solve the differential equation

$$y'' - y' = x.$$

1.9.2 INDEPENDENT VARIABLE MISSING

In case the variable x is missing from our differential equation, we make the substitution $y' = p$. This time the corresponding substitution for y'' will be a bit different. To wit,

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = \frac{dp}{dy} \cdot p.$$

This change of variable will reduce our differential equation to first-order. In the reduced equation, we treat p as the dependent variable (or function) and y as the independent variable.

EXAMPLE 1.18

Solve the differential equation

$$y'' + k^2 y = 0$$

[where it is understood that k is a real constant].

SOLUTION

We notice that the independent variable is missing. So we make the substitution

$$y' = p, \quad y'' = p \cdot \frac{dp}{dy}.$$

The equation then becomes

$$p \cdot \frac{dp}{dy} + k^2 y = 0.$$

In this new equation we can separate variables:

$$p \, dp = -k^2 y \, dy,$$

hence

$$\begin{aligned} \frac{p^2}{2} &= -k^2 \frac{y^2}{2} + C, \\ p &= \pm \sqrt{D - k^2 y^2} = \pm k \sqrt{E - y^2}. \end{aligned}$$

Now we re-substitute $p = dy/dx$ to obtain

$$\frac{dy}{dx} = \pm k \sqrt{E - y^2}.$$

We can separate variables to obtain

$$\frac{dy}{\sqrt{E - y^2}} = \pm k \, dx,$$

hence

$$\sin^{-1} \frac{y}{\sqrt{E}} = \pm kx + F$$

or

$$\frac{y}{\sqrt{E}} = \sin(\pm kx + F),$$

e.g.

thus

$$y = \sqrt{E} \sin(\pm kx + F).$$

Now we apply the sum formula for sine to rewrite the last expression as

$$y = \sqrt{E} \cos F \sin(\pm kx) + \sqrt{E} \sin F \cos(\pm kx).$$

A moment's thought reveals that we may consolidate the constants and finally write our general solution of the differential equation as

$$y = A \sin(kx) + B \cos(kx).$$

We shall learn in the next chapter a different, and perhaps more expeditious, method of attacking examples of the last type. It should be noted quite plainly in the last example, and also in some of the earlier examples of the section, that the method of reduction of order basically transforms the problem of solving one second-order equation to a new problem of solving *two* first-order equations. Examine each of the examples we have presented and see whether you can say what the two new equations are.

In the next example, we will solve a differential equation subject to an *initial condition*. This will be an important idea throughout the book. Solving a differential equation gives rise to a *family* of functions. Specifying the initial condition is a natural way to specialize down to a particular solution. In applications, these initial conditions will make good physical sense.

e.g.

EXAMPLE 1.19

Use the method of reduction of order to solve the differential equation

$$y'' = y' \cdot e^y$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

SOLUTION

We make the substitution

$$y' = p, \quad y'' = p \cdot \frac{dp}{dy}.$$

So the equation becomes

$$p \cdot \frac{dp}{dy} = p \cdot e^y.$$

We of course may separate variables, so the equation becomes

$$dp = e^y dy.$$

This is easily integrated to give

$$p = e^y + C.$$

Now we re-substitute $p = y'$ to find that

$$y' = e^y + C$$

or

$$\frac{dy}{dx} = e^y + C.$$

Because of the initial conditions $y(0) = 0$ and $[dy/dx](0) = 1$, we may conclude right away that $C = 0$. Thus our equation is

$$\frac{dy}{e^y} = dx$$

or

$$-e^{-y} = x + D.$$

Of course we can rewrite the equation finally as

$$y = -\ln(-x + E).$$

Since $y(0) = 0$, we conclude that

$$y(x) = -\ln(-x + 1)$$

is the solution of our initial value problem.

You Try It: Use the method of reduction of order to solve the differential equation 

$$y'' - y'y = 0.$$

You Try It: Use the method of reduction of order to solve the initial value problem 

$$y'' + y'y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

1.10.2 PURSUIT CURVES

A submarine speeds across the ocean bottom in a particular path, and a destroyer at a remote location decides to engage in pursuit. What path does the destroyer follow? Problems of this type are of interest in a variety of applications. We examine a few examples. The first one is purely mathematical, and devoid of “real world” trappings.

e.g.

EXAMPLE 1.20

A point P is dragged along the x - y plane by a string PT of fixed length a . If T begins at the origin and moves along the positive y -axis, and if P starts at the point $(a, 0)$, then what is the path of P ?

SOLUTION

The curve described by P is called, in the classical literature, a *tractrix* (from the Latin *tractum*, meaning “drag”). Figure 1.9 exhibits the salient features of the problem.

Observe that we can calculate the slope of the pursuit curve at the point P in two ways: (i) as the derivative of y with respect to x , and (ii) as the ratio of sides of the relevant triangle. This leads to the equation

$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}.$$

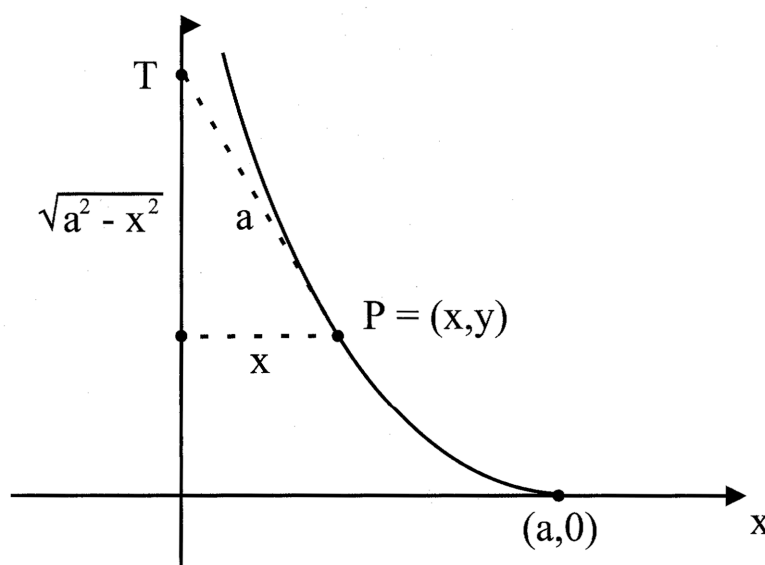


Fig. 1.9.

This is a separable, first-order differential equation. We write

$$\int dy = - \int \frac{\sqrt{a^2 - x^2}}{x} dx.$$

Performing the integrations (the right-hand side requires the trigonometric substitution $x = \sin \psi$), we find that

$$y = a \ln \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right) - \sqrt{a^2 - x^2}$$

is the equation of the tractrix.²

EXAMPLE 1.21

A rabbit begins at the origin and runs up the y -axis with speed a feet per second. At the same time, a dog runs at speed b from the point $(c, 0)$ in pursuit of the rabbit. What is the path of the dog?

e.g.

SOLUTION

At time t , measured from the instant both the rabbit and the dog start, the rabbit will be at the point $R = (0, at)$ and the dog at $D = (x, y)$. We wish to solve for y as a function of x . Refer to Fig. 1.10.

The premise of a pursuit analysis is that the line through D and R is tangent to the path—that is, the dog will always run straight at the rabbit. This immediately gives the differential equation

$$\frac{dy}{dx} = \frac{y - at}{x}.$$

This equation is a bit unusual for us, since x and y are both unknown functions of t . First, we rewrite the equation as

$$xy' - y = -at.$$

[Here the $'$ on y stands for differentiation in x .] We differentiate this equation with respect to x , which gives

$$xy'' = -a \frac{dt}{dx}.$$

²This curve is of considerable interest in other parts of mathematics. If it is rotated about the y -axis, then the result is a surface that gives a model for non-Euclidean geometry. The surface is called a *pseudosphere* in differential geometry. It is a surface of constant negative curvature (as opposed to a traditional sphere, which is a surface of constant positive curvature).

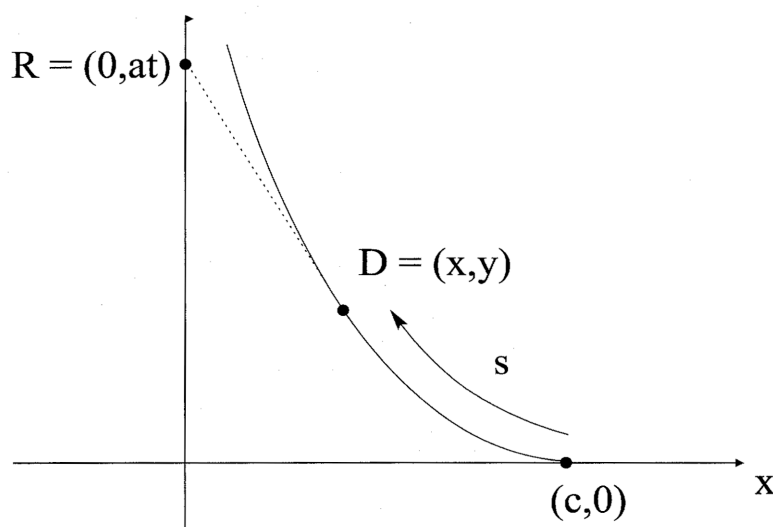


Fig. 1.10.

Since s is arc length along the path of the dog, it follows that $ds/dt = b$. Hence

$$\frac{dt}{dx} = \frac{dt}{ds} \cdot \frac{ds}{dx} = -\frac{1}{b} \cdot \sqrt{1 + (y')^2};$$

here the minus sign appears because s decreases when x increases (see Fig. 1.10). Combining the last two displayed equations gives

$$sy'' = \frac{a}{b} \sqrt{1 + (y')^2}.$$

For convenience, we set $k = a/b$, $y' = p$, and $y'' = dp/dx$ (the latter two substitutions being one of our standard reduction of order techniques). Thus we have

$$\frac{dp}{\sqrt{1 + p^2}} = k \frac{dx}{x}.$$

Now we may integrate, using the condition $p = 0$ when $x = c$. The result is

$$\ln \left(p + \sqrt{1 + p^2} \right) = \ln \left(\frac{x}{c} \right)^k.$$

When we solve for p , we find that

$$\frac{dy}{dx} = p = \frac{1}{2} \left[\left(\frac{x}{c} \right)^k - \left(\frac{c}{x} \right)^k \right].$$

In order to continue the analysis, we need to know something about the relative sizes of a and b . Suppose, for example, that $a < b$ (so $k < 1$), meaning that the dog will certainly catch the rabbit. Then we can integrate the last equation to obtain

$$y(x) = \frac{1}{2} \left[\frac{c}{k+1} \left(\frac{x}{c} \right)^{k+1} - \frac{c}{(1-k)} \left(\frac{c}{x} \right)^{k-1} \right] + D.$$

Since $y = 0$ when $x = c$, we find that $D = ck$. Of course the dog catches the rabbit when $x = 0$. Since both exponents on x are positive, we can set $x = 0$ and solve for y to obtain $y = ck$ as the point at which the dog and the rabbit meet.

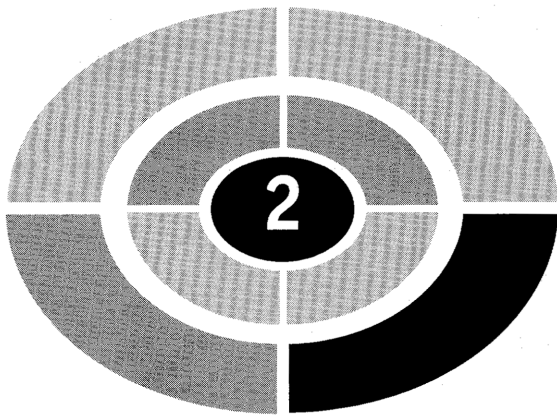
We invite you to consider what happens when $a = b$ and hence $k = 1$.

Math Note: The idea of and analysis of pursuit curves is of great interest to the navy. Battle strategies are devised using these ideas.



Exercises

- Verify that the following functions (explicit or implicit) are solutions of the corresponding differential equations:
 - $y = x^2 + c$ $y' = 2x$
 - $y = cx^2$ $xy' = 2y$
- Find the general solution of each of the following differential equations:
 - $y' = e^{3x} - x$
 - $y' = xe^{x^2}$
- For each of the following differential equations, find the particular solution that satisfies the given initial condition:
 - $y' = xe^x$ $y = 3$ when $x = 1$
 - $y' = 2 \sin x \cos x$ $y = 1$ when $x = 0$
- Use the method of separation of variables to solve each of these ordinary differential equations:
 - $x^5 y' - y^{-5} = 0$
 - $y' = 4xy$
- For each of the following differential equations, find the particular solution that satisfies the additional given property (called an *initial condition*):
 - $y'y = x + 1$ $y = 3$ when $x = 1$
 - $(dy/dx)x^2 = y$ $y = 2$ when $x = 1$
- Find the general solution of each of the following first-order, linear ordinary differential equations:
 - $y' - xy = 0$
 - $y' + 2xy = 2x$
- A tank contains 10 gallons of brine in which 2 pounds of salt are dissolved. New brine containing 1 pound of salt per gallon is pumped into the tank at the rate of 3 gallons per minute. The mixture is stirred and drained off at the rate of 4 gallons per minute. Find the amount $x = x(t)$ of salt in the tank at any time t .



CHAPTER

Second-Order Equations

2.1 Second-Order Linear Equations with Constant Coefficients

Second-order linear equations are important because they arise frequently in engineering and physics. For instance, acceleration is given by the second derivative, and force is mass times acceleration.

In this section we learn about *second-order linear equations with constant coefficients*. The “linear” attribute means, just as it did in the first-order situation, that the unknown function and its derivatives are not multiplied together, are not raised to powers, and are not the arguments of other function. So, for example,

$$y'' - 3y' + 6y = 0$$



is second-order linear while

$$\sin y'' - y' + 5y = 0$$

and

$$y \cdot y'' + 4y' + 3y = 0$$

are not.

The “constant coefficient” attribute means that the coefficients in the equation are not functions—they are constants. Thus a second-order linear equation with constant coefficient will have the form

$$ay'' + by' + cy = d, \quad (1)$$

where a, b, c, d are constants.

We in fact begin with the *homogeneous case*; this is the situation in which $d = 0$. We solve the equation (1) by a process of organized guessing: any solution of (1) will be a function that cancels with its derivatives. Thus it is a function that is similar in form to its derivatives. Certainly exponentials fit this description. Thus we guess a solution of the form

$$y = e^{rx}.$$

Plugging this guess into (1) gives

$$a[e^{rx}]'' + b[e^{rx}]' + c[e^{rx}] = 0.$$

Calculating the derivatives, we find that

$$a \cdot r^2 \cdot e^{rx} + b \cdot r \cdot e^{rx} + c \cdot e^{rx} = 0$$

or

$$[ar^2 + br + c] \cdot e^{rx} = 0.$$

This last equation can only be true (for all x) if

$$ar^2 + br + c = 0.$$

Of course this is a simple quadratic equation (called the *associated polynomial equation*), and we may solve it using the quadratic formula. This process will lead to our solution set.

EXAMPLE 2.1

Solve the differential equation

$$y'' - 5y' + 4y = 0.$$

e.g.

SOLUTION

Following the paradigm just outlined, we guess a solution of the form $y = e^{rx}$. This leads to the quadratic equation for r given by

$$r^2 - 5r + 4 = 0.$$

Of course this easily factors into

$$(r - 1)(r - 4) = 0,$$

so $r = 1, 4$.

Thus e^x and e^{4x} are solutions to the differential equation. A *general solution* is given by

$$y = A \cdot e^x + B \cdot e^{4x}, \quad (2)$$

where A and B are arbitrary constants. You may check that any function of the form (2) solves the original differential equation. Observe that our general solution (2) has two undetermined constants, which is consistent with the fact that we are solving a second-order differential equation.

**EXAMPLE 2.2**

Solve the differential equation

$$2y'' + 6y' + 2y = 0.$$

SOLUTION

The associated polynomial equation is

$$2r^2 + 6r + 2 = 0.$$

This equation does not factor in any obvious way, so we use the quadratic formula:

$$r = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = \frac{-6 \pm \sqrt{20}}{4} = \frac{-3 \pm \sqrt{5}}{2}.$$

Thus the general solution to the differential equation is

$$y = A \cdot e^{\frac{-3+\sqrt{5}}{2}x} + B \cdot e^{\frac{-3-\sqrt{5}}{2}x}.$$



Math Note: Much of the analysis that we have applied to second-order, constant coefficient, linear equations will apply, virtually without change, to constant coefficient, linear equations of high order. We shall say more about this topic in Section 2.7.

You Try It: Find the general solution of the second-order linear differential equation



$$y'' - 6y' + 5y = 0.$$

EXAMPLE 2.3

Solve the differential equation

e.g.

$$y'' - 6y' + 9y = 0.$$

SOLUTION

In this case the associated polynomial is

$$r^2 - 6r + 9 = 0.$$

This algebraic equation has the single solution $r = 3$. But our differential equation is second-order, and therefore we seek *two independent solutions*.

In the case that the associated polynomial has just one root, we find the other solution with an augmented guess: Our new guess is $y = x \cdot e^{3x}$. You may check for yourself that this new guess is also a solution. So the general solution of the differential equation is

$$y = A \cdot e^{3x} + B \cdot x e^{3x}.$$

You Try It: Find the general solution of the differential equation



$$y'' + 4y' + 4y = 0.$$

As a prologue to our next example, we must review some ideas connected with complex exponentials. Recall that, for a real variable x ,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

This equation persists if we replace the real variable x by a complex variable z . Thus

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{z^j}{j!}.$$

Now write $z = x + iy$, and let us gather together the real and imaginary parts of this last equation:

$$\begin{aligned}
 e^z &= e^{x+iy} \\
 &= e^x \cdot e^{iy} \\
 &= e^x \cdot \left[1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots \right] \\
 &= e^x \cdot \left[\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - + \cdots \right) + i \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - + \cdots \right) \right] \\
 &= e^x [\cos y + i \sin y].
 \end{aligned}$$

In the special case $x = 0$, the equation

$$e^{iy} = \cos y + i \sin y$$

is known as *Euler's formula*, in honor of Leonhard Euler (1707–1783). We will also make considerable use of the more general formula

$$e^{x+iy} = e^x [\cos y + i \sin y].$$

In using complex numbers, you should of course remember that the square root of a negative number is an *imaginary number*. For instance,

$$\sqrt{-4} = \pm 2i \quad \text{and} \quad \sqrt{-25} = \pm 5i.$$

e.g.

EXAMPLE 2.4

Solve the differential equation

$$4y'' + 4y' + 2y = 0.$$

SOLUTION

The associated polynomial is

$$4r^2 + 4r + 2 = 0.$$

We apply the quadratic equation to solve it:

$$r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4 \cdot 2}}{2 \cdot 4} = \frac{-4 \pm \sqrt{-16}}{8} = \frac{-1 \pm i}{2}.$$

Thus the solutions to our differential equation are

$$y = e^{\frac{-1+i}{2} \cdot x} \quad \text{and} \quad y = e^{\frac{-1-i}{2} \cdot x}.$$

A general solution is given by

$$y = A \cdot e^{\frac{-1+i}{2} \cdot x} + B \cdot e^{\frac{-1-i}{2} \cdot x}.$$

Using Euler's formula, we may rewrite this general solution as

$$\begin{aligned} y &= A \cdot e^{-x/2} [\cos x/2 + i \sin x/2] \\ &\quad + B e^{-x/2} [\cos x/2 - i \sin x/2]. \end{aligned} \quad (3)$$

We shall now use some propitious choices of A and B to extract meaningful real-valued solutions. First choose $A = 1/2$, $B = 1/2$. Putting these values in equation (3) gives

$$y = e^{-x/2} \cos x/2.$$

Now taking $A = -i/2$, $B = i/2$ gives the solution

$$y = e^{-x/2} \sin x/2.$$

As a result of this little trick, we may rewrite the general solution to our differential equation as

$$y = A \cdot e^{-x/2} \cos x/2 + B \cdot e^{-x/2} \sin x/2.$$

You Try It: Find the general solution of the differential equation



$$y'' + y' + y = 0.$$

Write this solution without using complex numbers (but certainly use complex numbers to *find* the solution).

Math Note: Complex numbers and complex analysis have a long history. For a long time these numbers were considered to be suspect—they did not really exist, but they had certain uses that made them tolerable. Today we know how to construct the complex numbers in a concrete manner (see [KRA2], [KRA4]).



We conclude this section with a last example of homogeneous, second-order, linear ordinary differential equation with constant coefficients, and with complex roots, just to show how straightforward the methodology really is.

EXAMPLE 2.5

Solve the differential equation



$$y'' - 2y' + 5y = 0.$$

SOLUTION

The associated polynomial is

$$r^2 - 2r + 5 = 0.$$

According to the quadratic formula, the solutions of this equation are

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Hence the roots of the associated polynomial are $r = 1 + 2i$ and $1 - 2i$.

According to what we have learned, two independent solutions to the differential equation are thus given by

$$y = e^x \cos 2x \quad \text{and} \quad y = e^x \sin 2x.$$

Therefore the general solution is given by

$$y = Ae^x \cos 2x + Be^x \sin 2x.$$



You Try It: Find the general solution of the differential equation

$$2y'' - 3y' + 6y = 0.$$

2.2 The Method of Undetermined Coefficients

“Undetermined coefficients” is a method of organized guessing. We have already seen guessing, in one form or another, serve us well in solving first-order linear equations and also in solving homogeneous second-order linear equations with constant coefficients. Now we shall expand the technique to cover *inhomogeneous* second-order linear equations.

We must begin by discussing what the solution to such an equation will look like. Consider an equation of the form

$$ay'' + by' + cy = f(x). \quad (1)$$

Suppose that we can find (by guessing or by some other means) a function $y = y_0(x)$ that satisfies this equation. We call y_0 a *particular solution* of the differential equation. Notice that it will *not* be the case that a constant multiple of y will also

solve the equation. In fact, if we consider $y = A \cdot y_0$ and plug this function into the equation, we obtain

$$a[Ay_0]'' + b[Ay_0]' + c[Ay_0] = A[ay_0'' + by_0' + cy_0] = A \cdot f.$$

But we expect the solution of a second-order equation to have two free constants. Where will they come from?

The answer is that we must separately solve the associated *homogeneous equation*, which is

$$ay'' + by' + cy = 0.$$

If y_1 and y_2 are solutions of this equation, then of course (as we learned in the last section) we know that $A \cdot y_1 + B \cdot y_2$ will be a general solution of *this homogeneous equation*. But then the general solution of the original differential equation (1) will be

$$y = y_0 + A \cdot y_1 + B \cdot y_2.$$

Math Note: We invite you to verify that, no matter what the choice of A and B , this y will be a solution of the original differential equation (1).



These ideas are best hammered home by the examination of some examples.

EXAMPLE 2.6

Find the general solution of the differential equation



$$y'' + y = \sin x. \quad (2)$$

SOLUTION

We might guess that $y = \sin x$ or $y = \cos x$ is a particular solution of this equation. But in fact these are solutions of the homogeneous equation

$$y'' + y = 0$$

(as we may check by using the techniques of the last section, or just by direct verification). So if we want to find a particular solution of (2), then we must try a bit harder.

Inspired by our experience with the case of repeated roots for the second-order, homogeneous linear equation with constant coefficients (as in the last section), we shall instead guess

$$y_0 = \alpha \cdot x \cos x + \beta \cdot x \sin x$$

for our particular solution. Notice that we allow arbitrary constants in front of the functions $x \cos x$ and $x \sin x$. These are the “undetermined coefficients” that we seek.

Now we simply plug the guess into the differential equation and see what happens. Thus

$$[\alpha \cdot x \cos x + \beta \cdot x \sin x]'' + [\alpha \cdot x \cos x + \beta \cdot x \sin x] = \sin x$$

or

$$\begin{aligned} &\alpha (2(-\sin x) + x(-\cos x)) + \beta (2 \cos x \\ &\quad + x(-\sin x)) + \alpha x \cos x + \beta x \sin x = \sin x \end{aligned}$$

or

$$(-2\alpha) \sin x + (2\beta) \cos x + (-\beta + \beta)x \sin x + (-\alpha + \alpha)x \cos x = \sin x.$$

We see that there is considerable cancellation, and we end up with

$$-2\alpha \sin x + 2\beta \cos x = \sin x.$$

The only way that this can be an identity in x is if $-2\alpha = 1$ and $2\beta = 0$ or $\alpha = -1/2$ and $\beta = 0$.

Thus our particular solution is

$$y_0 = -\frac{1}{2}x \cos x$$

and our general solution is

$$y = -\frac{1}{2}x \cos x + A \cos x + B \sin x.$$

e.g.

EXAMPLE 2.7

Find the solution of

$$y'' - y' - 2y = 4x^2$$

that satisfies $y(0) = 0$ and $y'(0) = 1$.

SOLUTION

The associated homogeneous equation is

$$y'' - y' - 2y = 0$$

and this has associated polynomial

$$r^2 - r - 2 = 0.$$

The roots are obviously $r = 2, -1$ and so the general solution of the associated homogeneous equation is $y = A \cdot e^{2x} + B \cdot e^{-x}$.

For a particular solution, our guess will be a polynomial. Guessing a second-degree polynomial makes good sense, since a guess of a higher-order polynomial is going to produce terms of high degree that we do not want. Thus we guess that $y_p(x) = \alpha x^2 + \beta x + \gamma$. Plugging this guess into the differential equation gives

$$[\alpha x^2 + \beta x + \gamma]'' - [\alpha x^2 + \beta x + \gamma]' - 2[\alpha x^2 + \beta x + \gamma] = 4x^2$$

or

$$[2\alpha] - [\alpha \cdot 2x + \beta] - [2\alpha x^2 + 2\beta x + 2\gamma] = 4x^2.$$

Grouping like terms together gives

$$-2\alpha x^2 + [-2\alpha - 2\beta]x + [2\alpha - \beta - 2\gamma] = 4x^2.$$

As a result, we find that

$$-2\alpha = 4$$

$$-2\alpha - 2\beta = 0$$

$$2\alpha - \beta - 2\gamma = 0.$$

This system is easily solved to yield $\alpha = -2$, $\beta = 2$, $\gamma = -3$. So our particular solution is $y_0(x) = -2x^2 + 2x - 3$. The general solution of the original differential equation is then

$$y(x) = (-2x^2 + 2x - 3) + A \cdot e^{2x} + B \cdot e^{-x}.$$

Now we seek the solution that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$. These translate to

$$0 = y(0) = -2 \cdot 0^2 + 2 \cdot 0 - 3 + A \cdot e^0 + B \cdot e^0$$

and

$$1 = y'(0) = -4 \cdot 0 + 2 - 0 + 2A \cdot e^0 - B \cdot e^0.$$

This gives the equations

$$0 = -3 + A + B$$

$$1 = 2 + 2A - B.$$

Of course we can solve this system quickly to find that $A = 1/3$, $B = 8/3$.

In conclusion, the solution to our initial boundary value problem is

$$y(x) = -2x^2 + 2x - 3 + \frac{1}{3} \cdot e^{2x} - \frac{8}{3} \cdot e^{-x}.$$



You Try It: Solve the differential equation

$$y'' - y = \cos x.$$



You Try It: Find the solution to the initial value problem

$$y'' + y' = x, \quad y(0) = 1, \quad y'(0) = 0.$$



Math Note: If we wish to use the method of undetermined coefficients to solve the differential equation

$$y^{(iv)} + 2y^{(ii)} + y = \sin x,$$

then we must note that $\sin x$, $\cos x$, $x \sin x$, and $x \cos x$ are all solutions of the associated homogeneous equation

$$y^{(iv)} + 2y^{(ii)} + y = 0.$$

Thus we will need to guess a particular solution of the form $Ax^2 \cos x + Bx^2 \sin x$. We invite the reader to try this guess and find a particular solution.

2.3 The Method of Variation of Parameters

Variation of parameters is a method for producing a *particular solution* to an inhomogeneous equation by exploiting the (usually much simpler to find) solutions to the associated homogeneous equation.

Let us consider the differential equation

$$y'' + p(x)y' + q(x)y = r(x). \quad (1)$$

Assume that, by some method or other, we have found the general solution of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

to be

$$y = Ay_1(x) + By_2(x).$$

What we do now is to *guess* that a particular solution to the original equation (1) has the form

$$y_0(x) = v_1(x) \cdot y_1(x) + v_2(x) \cdot y_2(x). \quad (2)$$

Now let us proceed on this guess. We calculate that

$$y'_0 = [v'_1 y_1 + v_1 y'_1] + [v'_2 y_1 + v_2 y'_2] = [v'_1 y_1 + v'_2 y_2] + [v_1 y'_1 + v_2 y'_2]. \quad (3)$$

Now we also need to calculate the second derivative of y_0 . But we do not want the extra complication of having second derivatives of v_1 and v_2 . So we will mandate that the first expression in brackets on the far right side of (3) is identically zero. Thus we have

$$v'_1 y_1 + v'_2 y_2 \equiv 0. \quad (4)$$

Thus

$$y'_0 = v_1 y'_1 + v_2 y'_2$$

and we can now calculate that

$$y''_0 = [v'_1 y'_1 + v_1 y''_1] + [v'_2 y'_2 + v_2 y''_2]. \quad (5)$$

Now let us substitute (2), (3), and (5) into the differential equation. The result is

$$\begin{aligned} & ([v'_1 y'_1 + v_1 y''_1] + [v'_2 y'_2 + v_2 y''_2]) + p(x) \cdot (v_1 y'_1 + v_2 y'_2) \\ & + q(x) \cdot (v_1 y_1 + v_2 y_2). \end{aligned}$$

After some algebraic manipulation, this becomes

$$v_1 (y''_1 + p y'_1 + q y_1) + v_2 (y''_2 + p y'_2 + q y_2) + v'_1 y'_1 + v'_2 y'_2 = r.$$

Since y_1, y_2 are solutions of the homogeneous equation, the expressions in parentheses vanish. The result is

$$v'_1 y'_1 + v'_2 y'_2 = r. \quad (6)$$

At long last we have two equations to solve in order to determine what v_1 and v_2 must be. Namely, we use equations (4) and (6) to obtain

$$v'_1 y_1 + v'_2 y_2 = 0,$$

$$v'_1 y'_1 + v'_2 y'_2 = r.$$

In practice, these can be often solved for v'_1, v'_2 , and then integration tells us what v_1, v_2 must be.

As usual, the best way to understand a new technique is by way of some examples.

EXAMPLE 2.8

Find the general solution of

$$y'' + y = \csc x.$$

e.g.

SOLUTION

Of course the general solution to the associated homogeneous equation is familiar. It is

$$y(x) = A \sin x + B \cos x.$$

In order to find a particular solution, we need to solve the equations

$$v_1' \sin x + v_2' \cos x = 0,$$

$$v_1'(\cos x) + v_2'(-\sin x) = \csc x.$$

This is a simple algebra problem, and we find that

$$v_1'(x) = \cot x \quad \text{and} \quad v_2'(x) = -1.$$

As a result,

$$v_1(x) = \ln(\sin x) \quad \text{and} \quad v_2(x) = -x.$$

[As you will see, we do not need any constants of integration.]

The final result is then that a particular solution of our differential equation is

$$y_0(x) = v_1(x)y_1(x) + v_2(x)y_2(x) = [\ln(\sin x)] \cdot \sin x + [-x] \cdot \cos x.$$

We invite you to check that this solution actually works. The general solution of the original differential equation is

$$y(x) = ([\ln(\sin x)] \cdot \sin x + [-x] \cdot \cos x) + A \sin x + B \cos x.$$

**EXAMPLE 2.9**

Solve the differential equation

$$y'' - y' - 2y = 4x^2$$

using the method of variation of parameters.

SOLUTION

You will note that, in the last section (Example 2.7), we solved this same equation using the method of undetermined coefficients (or organized guessing). Now we will solve it a second time by our new method.

As we saw before, the homogeneous equation has the general solution

$$y = Ae^{2x} + Be^{-x}.$$

Now we solve the system

$$\begin{aligned}v_1'e^{2x} + v_2'e^{-x} &= 0, \\v_1'[2e^{2x}] + v_2'[-e^{-x}] &= 4x^2.\end{aligned}$$

The result is

$$v_1'(x) = \frac{4}{3}x^2e^{-2x} \quad \text{and} \quad v_2'(x) = -\frac{4}{3}x^2e^x.$$

We may use integration by parts to then determine that

$$v_1(x) = -\frac{2x^2}{3}e^{-2x} - \frac{2x}{3}e^{-2x} - \frac{1}{3}e^{-2x}$$

and

$$v_2(x) = -\frac{4x^2}{3}e^x + \frac{8x}{3}e^x - \frac{8}{3}e^x.$$

We finally see that a particular solution to our differential equation is

$$\begin{aligned}y_0(x) &= v_1(x) \cdot y_1(x) + v_2(x)y_2(x) \\&= \left[-\frac{2x^2}{3}e^{-2x} - \frac{2x}{3}e^{-2x} - \frac{1}{3}e^{-2x}\right] \cdot e^{2x} \\&\quad + \left[-\frac{4x^2}{3}e^x + \frac{8x}{3}e^x - \frac{8}{3}e^x\right] \cdot e^{-x} \\&= \left[-\frac{2x^2}{3} - \frac{2x}{3} - \frac{1}{3}\right] + \left[-\frac{4x^2}{3} + \frac{8x}{3} - \frac{8}{3}\right] \\&= -2x^2 + 2x - 3.\end{aligned}$$

In conclusion, the general solution of the original differential equation is

$$y(x) = (-2x^2 + 2x - 3) + Ae^{2x} + Be^{-x}.$$

As you can see, this is the same answer that we obtained in Section 2.2, Example 2.7, by the method of undetermined coefficients.

You Try It: Use the method of variation of parameters to find the general solution of the differential equation



$$y'' - 2y = x + 1.$$

EQUATION LINÉAIRE DU 2^e ORDRE À COEFFICIENTS CONSTANTS : RÉSUMÉ

L'équation est du type $ay'' + by' + cy = f(x)$, avec $a \neq 0$.

1. Cas où $f(x) = 0$ (équation sans second membre ou homogène)

La solution dépend de l'équation caractéristique $ar^2 + br + c = 0$:

Si l'équation caractéristique possède...	la solution de l'équation différentielle est...
deux solutions réelles r_1 et r_2	$y = Ae^{r_1x} + Be^{r_2x}$
une solution r	$y = (Ax + B)e^{rx}$
deux solutions complexes $p \pm qi$	$y = e^{px} (A \cos(qx) + B \sin(qx))$

2. Cas général

La solution générale est la somme d'une solution particulière et de la solution générale de l'équation sans second membre $ay'' + by' + cy = 0$.

Pour trouver une solution particulière, on tient compte de la forme du second membre $f(x)$ en se référant au tableau ci-dessous.

Lorsque $f(x)$ est du type...	on cherche une solution particulière du type...
polynôme de degré n	polynôme de degré n si $c \neq 0$ polynôme de degré $n+1$ si $c = 0$ et $b \neq 0$ polynôme de degré $n+2$ si $b = c = 0$
λe^{kx}	αe^{kx} ou $\alpha x e^{kx}$ ou $\alpha x^2 e^{kx}$
$\lambda \sin(\omega x)$	$\alpha \sin(\omega x) + \beta \cos(\omega x)$ ou $\alpha x \cos(\omega x)$
$\lambda \cos(\omega x)$	$\alpha \sin(\omega x) + \beta \cos(\omega x)$ ou $\alpha x \sin(\omega x)$
$e^{kx} (\lambda \sin(\omega x) + \mu \cos(\omega x))$	$e^{kx} (\alpha \sin(\omega x) + \beta \cos(\omega x))$ ou $x e^{kx} (\alpha \sin(\omega x) + \beta \cos(\omega x))$
combinaison linéaire des types précédents	combinaison linéaire des solutions particulières proposées pour ces différents types

$$(k, \alpha, \beta, \lambda, \mu, \omega \in \mathbb{R})$$

2.5 Vibrations and Oscillations

When a physical system in stable equilibrium is disturbed, then it is subject to forces that tend to restore the equilibrium. The result is a system that can lead to oscillations or vibrations. It is described by an ordinary differential equation of the form

$$\frac{d^2x}{dt^2} + p(t) \cdot \frac{dx}{dt} + q(t)x = r(t).$$

In this section we shall learn how and why such an equation models the physical system we have described, and we shall see how its solution sheds light on the physics of the situation.

2.5.1 UNDAMPED SIMPLE HARMONIC MOTION

Our basic example will be a cart of mass M attached to a nearby wall by means of a spring. See Fig. 2.1. The spring exerts no force when the cart is at its rest position $x = 0$. According to Hooke's Law, if the cart is displaced a distance x , then the spring exerts a proportional force $F_s = -kx$, where k is a positive constant known as Hooke's constant. Observe that if $x > 0$ then the cart is moved to the right and the spring pulls to the left; so the force is negative. Obversely, if $x < 0$ then the cart is moved to the left and the spring resists with a force to the right; so the force is positive.

Newton's second law of motion says that the mass of the cart times its acceleration equals the force acting on the cart. Thus

$$M \cdot \frac{d^2x}{dt^2} = F_s = -k \cdot x.$$

As a result,

$$\frac{d^2x}{dt^2} + \frac{k}{M}x = 0.$$

It is both convenient and traditional to let $a = \sqrt{k/M}$ (both k and M are positive) and thus to write the equation as

$$\frac{d^2x}{dt^2} + a^2x = 0.$$

Of course this is a familiar differential equation for us, and we can write its general solution immediately:

$$x(t) = A \sin at + B \cos at.$$

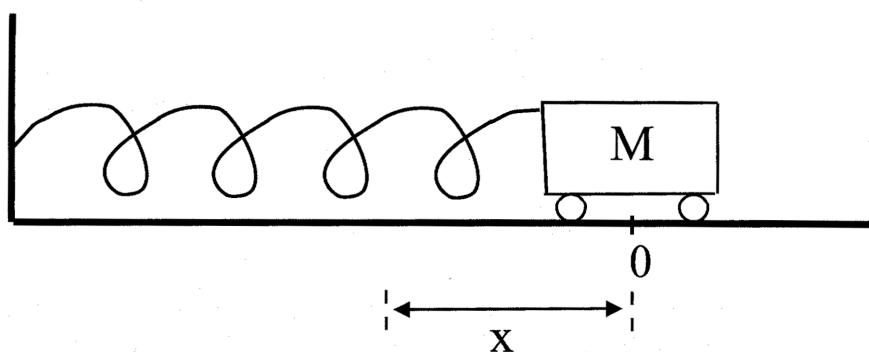


Fig. 2.1.

Now suppose that the cart is pulled to the right to an initial position of $x = x_0$ and then is simply released (with initial velocity 0). Then we have the initial conditions

$$x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(0) = 0.$$

Thus

$$\begin{aligned} x_0 &= A \sin(a \cdot 0) + B \cos(a \cdot 0) \\ 0 &= Aa \cos(a \cdot 0) - Ba \sin(a \cdot 0) \end{aligned}$$

or

$$\begin{aligned} x_0 &= B \\ 0 &= A \cdot a. \end{aligned}$$

We conclude that $B = x_0$, $A = 0$, and we find the solution of the system to be

$$x(t) = x_0 \cos at.$$

In other words, if the cart is displaced a distance x_0 and released, then the result is a simple harmonic motion (described by the cosine function) with *amplitude* x_0 (i.e., the cart glides back and forth, x_0 units to the left of the origin and then x_0 units to the right) and with period $T = 2\pi/a$ (which means that the motion repeats itself every $2\pi/a$ units of time).

The *frequency* f of the motion is the number of cycles per unit of time, hence $f \cdot T = 1$, or $f = 1/T = a/(2\pi)$. It is useful to substitute back in the actual value of a so that we can analyze the physics of the system. Thus

$$\text{Amplitude} = x_0$$

$$\text{Period} = T = \frac{2\pi\sqrt{M}}{\sqrt{k}}$$

$$\text{Frequency} = f = \frac{\sqrt{k}}{2\pi\sqrt{M}}.$$

Math Note: We see that if the stiffness of the spring k is increased then the period becomes smaller and the frequency increases. Likewise, if the mass of the cart is increased then the period increases and the frequency decreases. Thus the mathematical analysis coincides with, and reinforces, our physical intuition.



2.5.2 DAMPED VIBRATIONS

It probably has occurred to you that the physical model in the last subsection is not realistic. Typically, a cart that is attached to a spring and released, just as we have described, will enter a harmonic motion that *dies out over time*. In other words, resistance and friction will cause the system to be damped. Let us add that information to the system.

Physical considerations make it plausible to postulate that the resistance is proportional to the velocity of the moving cart. Thus the resistive force is

$$F_d = -c \frac{dx}{dt},$$

where F_d denotes damping force and $c > 0$ is a positive constant that measures the resistance of the medium (air or water or oil, etc.). Notice, therefore, that when the cart is traveling to the right, then $dx/dt > 0$ and therefore the force of resistance is negative (i.e., in the other direction). Likewise, when the cart is traveling to the left, then $dx/dt < 0$ and the force of resistance is positive.

Since the total of all the forces acting on the cart equals the mass times the acceleration, we now have

$$M \cdot \frac{d^2x}{dt^2} = F_s + F_d.$$

In other words,

$$\frac{d^2x}{dt^2} + \frac{c}{M} \cdot \frac{dx}{dt} + \frac{k}{M} \cdot x = 0.$$

Because of convenience and tradition, we again take $a = \sqrt{k/M}$ and we set $b = c/(2M)$. Thus the differential equation takes the form

$$\frac{d^2x}{dt^2} + 2b \cdot \frac{dx}{dt} + a^2 \cdot x = 0.$$

This is a second-order, linear, homogeneous ordinary differential equation with constant coefficients. The associated polynomial is

$$r^2 + 2br + a^2 = 0,$$

and it has roots

$$r_1, r_2 = \frac{-2b \pm \sqrt{4b^2 - 4a^2}}{2} = -b \pm \sqrt{b^2 - a^2}.$$

Now we must consider three cases.

Case A: $b^2 - a^2 > 0$ In words, we are assuming that the frictional force (which depends on c) is significantly larger than the stiffness of the spring (which depends on k). Thus we would expect the system to damp heavily. In any event, the calculation of r_1, r_2 involves the square root of a *positive real number*, and thus r_1, r_2 are distinct real (and negative) roots of the associated polynomial equation.

Thus the general solution of our system in this case is

$$x = Ae^{r_1 t} + Be^{r_2 t},$$

where (we repeat) r_1, r_2 are negative real numbers. We apply the initial conditions $x(0) = x_0$, $dx/dt(0) = 0$, just as in the last section (details are left to you). The result is the particular solution

$$x(t) = \frac{x_0}{r_1 - r_2} (r_1 e^{r_2 t} - r_2 e^{r_1 t}). \quad (1)$$

Notice that, in this heavily damped system, no oscillation occurs (i.e., there are no sines or cosines in the expression for $x(t)$). The system simply dies out. Figure 2.2 exhibits the graph of the function in (1).

Math Note: The type of harmonic motion illustrated in this last discussion, and in Fig. 2.2, is the ideal motion that is induced by the resistance of a shock absorber on an automobile. The whole purpose of a shock absorber is to make the harmonic motion, that would be induced by the car hitting a bump, die out immediately.



Case B: $b^2 - a^2 = 0$ This is the critical case, where the resistance balances the force of the spring. We see that $b = a$ (both are known to be positive) and $r_1 = r_2 = -b = -a$. We know, then, that the general solution to our differential

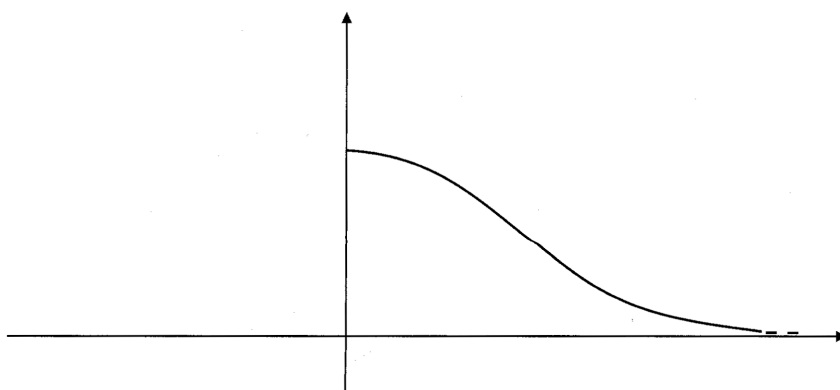


Fig. 2.2.

equation is

$$x(t) = Ae^{-at} + Bte^{-at}.$$

When the standard initial conditions are imposed, we find the particular solution

$$x(t) = x_0 \cdot e^{-at}(1 + at).$$

We see that this differs from the situation in Case A by the factor $(1 + at)$. That factor of course attenuates the damping, but there is *still no oscillatory motion*. We call this the *critical case*. The graph of our new $x(t)$ is quite similar to the graph already shown in Fig. 2.2.



Math Note: When a shock absorber begins to wear out, it becomes less effective. At a certain critical stage its action will be described more accurately by Case B than by Case A. In physical terms, this will mean that the oscillations of the car (induced by a road bump, for instance) will be damped out less effectively. The car will return to true more slowly.

If there is any small decrease in the viscosity, however slight, then the system will begin to vibrate (as one would expect). That is the next, and last, case that we examine.

Case C: $b^2 - a^2 < 0$ Now $0 < b < a$ and the damping is less than the force of the spring. The calculation of r_1, r_2 entails taking the square root of a negative number. Thus r_1, r_2 are the conjugate complex numbers $-b \pm i\sqrt{a^2 - b^2}$. We set $\alpha = \sqrt{a^2 - b^2}$.

Now the general solution of our system, as we well know, is

$$x(t) = e^{-bt}(A \sin \alpha t + B \cos \alpha t).$$

If we evaluate A, B according to our usual initial conditions, then we find the particular solution

$$x(t) = \frac{x_0}{\alpha} e^{-bt}(b \sin \alpha t + \alpha \cos \alpha t).$$

It is traditional and convenient to set $\theta = \tan^{-1}(b/\alpha)$. With this notation, we can express the last equation in the form

$$x(t) = \frac{x_0 \sqrt{\alpha^2 + b^2}}{\alpha} e^{-bt} \cos(\alpha t - \theta). \quad (2)$$

As you can see, there is oscillation because of the presence of the cosine function. The amplitude (the expression that appears in front of cosine) clearly

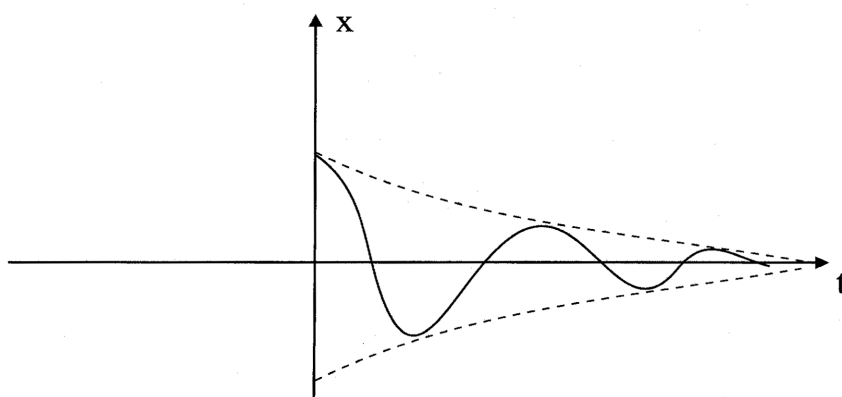


Fig. 2.3.

falls off—rather rapidly—with t because of the presence of the exponential. The graph of this function is exhibited in Fig. 2.3.

Of course this function is *not* periodic—it is dying off, and not repeating itself. What *is true*, however, is that the graph crosses the t -axis (the equilibrium position $x = 0$) at regular intervals. If we consider this interval T (which is not a “period,” strictly speaking) as the time required for one complete cycle, then $\alpha T = 2\pi$, so

$$T = \frac{2\pi}{\alpha} = \frac{2\pi}{\sqrt{k/M - c^2/(4M^2)}}. \quad (3)$$

We define the number f , which plays the role of “frequency” with respect to the indicated time interval, to be

$$f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{M} - \frac{c^2}{4M^2}}.$$

This number is commonly called the *natural frequency* of the system. When the viscosity vanishes, then our solution clearly reduces to the one we found earlier when there was no viscosity present. We also see that the frequency of the vibration is reduced by the presence of damping; increasing the viscosity further reduces the frequency.

Math Note: When the shock absorbers on your car are really shot, then the motion of the car after striking a bump will resemble that shown in Fig. 2.3. The harmonic motion begun by the bump will die out—but rather slowly. The result is discomfort for the passengers.

